

DERIVED CATEGORIES OF RESOLUTIONS OF CYCLIC QUOTIENT SINGULARITIES

ANDREAS KRUG, DAVID PLOOG, AND PAWEŁ SOSNA

ABSTRACT. For a cyclic group G acting on a smooth variety X with only one character occurring in the G -equivariant decomposition of the normal bundle of the fixed point locus, we study the derived categories of the orbifold $[X/G]$ and the blow-up resolution $\tilde{Y} \rightarrow X/G$.

Some results generalise known facts about $X = \mathbb{A}^n$ with diagonal G -action, while other results are new also in this basic case. In particular, if the codimension of the fixed point locus equals $|G|$, we study the induced tensor products under the equivalence $D^b(\tilde{Y}) \cong D^b([X/G])$ and give a 'flop-flop=twist' type formula. We also introduce candidates for general constructions of categorical crepant resolutions inside the derived category of a given geometric resolution of singularities and test these candidates on cyclic quotient singularities.

CONTENTS

1. Introduction	1
2. Preliminaries	4
3. The geometric setup	12
4. Proof of the main result	15
5. Categorical resolutions	25
6. Stability conditions for Kummer threefolds	31
References	32

1. INTRODUCTION

For geometric, homological and other reasons, it has become commonplace to study the bounded derived category of a variety. One of the many intriguing aspects are connections, some of them conjectured, some of them proven, to birational geometry.

One expected phenomenon concerns a birational correspondence

$$\begin{array}{ccc} & Z & \\ q \swarrow & & \searrow p \\ X & \text{-----} & X' \end{array}$$

of smooth varieties. Then we should have:

- A fully faithful embedding $D^b(X) \hookrightarrow D^b(X')$, if $q^*K_X \leq p^*K_{X'}$.
- A fully faithful embedding $D^b(X') \hookrightarrow D^b(X)$, if $q^*K_X \geq p^*K_{X'}$.
- An equivalence $D^b(X') \cong D^b(X)$, in the *flop* case $q^*K_X = p^*K_{X'}$.

This is proven in many instances; see [BO95], [Bri02], [Kaw02], [Nam03].

Another very interesting aspect of derived categories is their occurrence in the context of the McKay correspondence. Here, one of the key expectations is that the derived category

of a crepant resolution $\tilde{Y} \rightarrow X/G$ of a Gorenstein quotient variety is derived equivalent to the corresponding quotient orbifold: $D^b(\tilde{Y}) \cong D^b([X/G]) = D_G^b(X)$. In [BKR01], this expectation is proven in many cases under the additional assumption that $\tilde{Y} \cong \text{Hilb}^G(X)$ is the fine moduli space of G -clusters on X . It is enlightening to view the derived McKay correspondence as an orbifold version of the conjecture on derived categories under birational correspondences described above; for more information on this point of view, see [Kaw16, Sect. 2], where the conjecture is called the *DK-Hypothesis*. Indeed, if we denote the universal family of G -clusters by $\mathcal{Z} \subset \tilde{Y} \times X$, we have the following diagram of birational morphisms of orbifolds

$$(1) \quad \begin{array}{ccc} & [\mathcal{Z}/G] & \\ q \swarrow & & \searrow p \\ \tilde{Y} & \overset{\text{---}}{\longrightarrow} & [X/G] \\ \varrho \searrow & & \swarrow \pi \\ & X/G & \end{array}$$

Since the pullback of the canonical sheaf of X/G under π is the canonical sheaf of $[X/G]$, the condition that ϱ is a crepant resolution amounts to saying that (1) is a flop of orbifolds.

In many situations, a crepant resolution of X/G does not exist. However, given a resolution $\varrho: \tilde{Y} \rightarrow X$, the DK-Hypothesis still predicts the behavior of the categories $D^b(\tilde{Y})$ and $D_G^b(X)$ if $\varrho^*K_{X/G} \geq K_{\tilde{Y}}$ or $\varrho^*K_{X/G} \leq K_{\tilde{Y}}$. Another related idea is that, even though a crepant resolution does not exist in general, there should always be a *categorical crepant resolution* of $D^b(X/G)$; see [Kuz08]. The hope is to find such a categorical resolution as an admissible subcategory of the derived category $D^b(\tilde{Y})$ of a geometric resolution.

Besides dimensions 2 and 3, one of the most studied testing grounds for the above, and related, ideas is the isolated quotient singularity \mathbb{A}^n/μ_m . Here, the cyclic group μ_m of order m acts on the affine space by multiplication with a primitive m -th root of unity ζ . In this paper, we consider the following straight-forward generalisation of this set-up. Namely, let X be a quasi-projective smooth complex variety acted upon by the finite cyclic group μ_m . We assume that only 1 and μ_m occur as the isotropy groups of the action and write $S := \text{Fix}(\mu_m) \subset X$ for the fixed point locus. Fix a generator g of μ_m and assume that g acts on the normal bundle $N := N_{S/X}$ by multiplication with some fixed primitive m -th root of unity ζ . Then the blow-up $\tilde{Y} \rightarrow X/\mu_m$ with center S is a resolution of singularities; see Section 3 for further details. The four particular cases we have in mind are

- (a) $X = \mathbb{A}^n$ with the diagonal action of any μ_m .
- (b) $X = Z^2$, where Z is a smooth projective variety of arbitrary dimension, and $\mu_2 = \mathfrak{S}_2$ acts by permuting the factors. Then $\tilde{Y} \cong Z^{[2]}$, the Hilbert scheme of two points.
- (c) X is an abelian variety, μ_2 acts by ± 1 . In this case, \tilde{Y} is known as the *Kummer resolution*.
- (d) $X \rightarrow Y = X/\mu_m$ is a cyclic covering of a smooth variety Y , branched over a divisor. Here, $n = 1$ and $\tilde{X} = X$, $\tilde{Y} = Y$. This case has been studied in [KP14].

First, we prove the following result in Subsection 3.1. This is probably well-known to experts, but we could not find it in the literature. Write $G := \mu_m$.

Proposition 1.1. *The resolution obtained by blowing up the fixed point locus in X/G is isomorphic to the G -Hilbert scheme: $\tilde{Y} \cong \text{Hilb}^G(X)$.*

We set $n := \text{codim}(S \hookrightarrow X)$ and find the following dichotomy, in accordance with the DK-Hypothesis. We keep the notation from diagram (1). In particular, for $n = m$, we obtain new instances of BKR-style derived equivalences between orbifold and resolution.

Theorem 1.2.

- (i) *The functor $\Phi := p_*q^*: D^b(\tilde{Y}) \rightarrow D_G^b(X)$ is fully faithful for $m \geq n$ and an equivalence for $m = n$. For $m > n$, there is a semi-orthogonal decomposition of $D_G^b(X)$ consisting of $\Phi(D^b(\tilde{Y}))$ and $m - n$ pieces equivalent to $D^b(S)$.*
- (ii) *The functor $\Psi := q_*p^*: D_G^b(X) \rightarrow D^b(\tilde{Y})$ is fully faithful for $n \geq m$ and an equivalence for $n = m$. For $n > m$, there is a semi-orthogonal decomposition of $D^b(\tilde{Y})$ consisting of $\Psi(D_G^b(X))$ and $n - m$ pieces equivalent to $D^b(S)$.*

For a more exact statement with an explicit description of the embeddings of the $D^b(S)$ components into $D^b(\tilde{Y})$ and $D_G^b(X)$, see Section 4. In particular, for $m > n$, the push-forward $a_*: D^b(S) \rightarrow D_G^b(X)$ along the embedding $a: S \hookrightarrow X$ of the fixed point divisor is fully faithful.

In the basic affine case (a), the result of the theorem is also stated in [Kaw16, Ex. 4]. Proofs, in this basic case, are given in [Abu16, Sect. 4] for $n \geq m$ and in [IU15] for $n = 2$. If $n = 1$, the quotient is already smooth and we have $\tilde{Y} = X/G$ — here the semi-orthogonal decomposition categorifies the natural decomposition of the orbifold cohomology; compare [PVdB15]. The $n = 1$ case is also proven in [Lim16, Thm. 3.3.2].

We study the case $m = n$, where Φ and Ψ are equivalences, in more detail. On both sides of the equivalence, we have distinguished line bundles. The line bundle $\mathcal{O}_{\tilde{Y}}(Z)$ on \tilde{Y} , corresponding to the exceptional divisor, admits an m -th root \mathcal{L} . On $[X/G]$, there are twists of the trivial line bundle by the group characters $\mathcal{O}_X \otimes \chi^i$. For $i = -m + 1, \dots, -1, 0$, we have $\Psi(\mathcal{O}_X \otimes \chi^i) \cong \mathcal{L}^i$. Furthermore, we see that the functors $D^b(S) \rightarrow D^b(\tilde{Y})$ and $D^b(S) \rightarrow D_G^b(X)$, which define fully faithful embeddings in the $n > m$ and $m < n$ cases, respectively, become spherical for $m = n$ and hence induce twist autoequivalences; see Subsection 2.8 for details on spherical functors and twists. We show that the tensor products by the distinguished line bundles correspond to the spherical twists under the equivalences Ψ and Φ . In particular, one part of Theorem 4.26 is the following formula.

Theorem 1.3. *There is an isomorphism $\Psi^{-1}(\Psi(_) \otimes \mathcal{L}^{-1}) \cong T_{a_*}^{-1}(_ \otimes \chi^{-1})$ of autoequivalences of $D_G^b(X)$ where the inverse spherical twist $T_{a_*}^{-1}$ is defined by the exact triangle of functors*

$$T_{a_*}^{-1} \rightarrow \text{id} \rightarrow a_*(a^*(_)^G) \rightarrow .$$

The tensor powers of the line bundle \mathcal{L} form a strong generator of $D^b(\tilde{Y})$, thus Theorem 1.3, at least theoretically, completely describes the tensor product

$$_ \hat{\otimes} _ := \Psi^{-1}(\Psi(_) \otimes \Psi(_)): D_G^b(X) \times D_G^b(X) \rightarrow D_G^b(X)$$

induced by Ψ on $D_G^b(X)$. There is related unpublished work on induced tensor products under the McKay correspondence in dimensions 2 and 3 by T. Abdelgadir, A. Craw, J. Karmazyn, and A. King. In Corollary 4.27, we also get a formula which can be seen as a stacky instance of the 'flop-flop = twist' principle as discussed in [ADM15].

In Section 5, we introduce a general candidate for a weakly crepant categorical resolution (see [Kuz08] or Subsection 5.1 for this notion), namely the *weakly crepant neighbourhood*

$\mathrm{WCN}(\varrho) \subset \mathrm{D}^b(\tilde{Y})$, inside the derived category of a given resolution $\varrho: \tilde{Y} \rightarrow Y$ of a rational Gorenstein variety Y . The idea is pretty simple: by Grothendieck duality, there is a canonical section $s: \mathcal{O}_{\tilde{Y}} \rightarrow \mathcal{O}_{\varrho}$ of the relative dualising sheaf, and this induces a morphism of Fourier–Mukai transforms $t := \varrho_*(_ \otimes s): \varrho_* \rightarrow \varrho_!$.

Set $\varrho_+ := \mathrm{cone}(t)$ and $\mathrm{WCN}(\varrho) := \ker(\varrho_+)$. Then, by the very construction, we have $\varrho_*[\mathrm{WCN}(\varrho)] \cong \varrho_![\mathrm{WCN}(\varrho)]$ which amounts to the notion of categorical weak crepancy. The only thing open to ensure that $\mathrm{WCN}(\varrho)$ is a categorical weakly crepant resolution is whether it is actually a smooth category; this holds as soon as it is an admissible subcategory of $\mathrm{D}^b(\tilde{Y})$ which means that its inclusion has adjoints. We prove that, in the Gorenstein case $m \mid n$ of our set-up of cyclic quotients, $\mathrm{WCN}(\varrho) \subset \mathrm{D}^b(\tilde{Y})$ is an admissible subcategory; see [Theorem 5.4](#).

In [Subsection 5.4](#), we observe that there are various weakly crepant resolutions inside $\mathrm{D}^b(\tilde{Y})$. However, a strongly crepant categorical resolution inside $\mathrm{D}^b(\tilde{Y})$ is unique, as we show in [Proposition 5.9](#). Our concept of weakly crepant neighbourhoods was motivated by the idea that some non-CY objects possess ‘CY neighbourhoods’ (a construction akin to the spherical subcategories of spherelike objects in [\[HKP12\]](#)), i.e. full subcategories in which they become Calabi–Yau. This relationship is explained in [Subsection 5.5](#).

In the final [Section 6](#), we construct Bridgeland stability conditions on Kummer threefolds as an application of our results; see [Corollary 6.2](#).

Conventions. We work over the complex numbers. All functors are assumed to be derived. We write $\mathcal{H}^i(E)$ for the i -th cohomology object of a complex $E \in \mathrm{D}^b(Z)$ and $H^*(E)$ for the complex $\oplus_i H^i(Z, E)[-i]$. If a functor Φ has a left/right adjoint, they are denoted Φ^L, Φ^R .

There are a number of spaces, maps and functors repeatedly used in this text. For the convenience of the reader, we collect our notation at the very end of this article, on [page 33](#).

Acknowledgements. It is a pleasure to thank Tarig Abdelgadir, Martin Kalck, Sönke Rollenske and Evgeny Shinder for comments and discussions.

2. PRELIMINARIES

2.1. Fourier–Mukai transforms and kernels. Recall that given an object \mathcal{E} in $\mathrm{D}^b(Z \times Z')$, where Z and Z' are smooth and projective, we get an exact functor $\mathrm{D}^b(Z) \rightarrow \mathrm{D}^b(Z')$, $F \mapsto p_{Z'}^*(\mathcal{E} \otimes p_Z^* F)$. Such a functor, denoted by $\mathrm{FM}_{\mathcal{E}}$, is called a *Fourier–Mukai transform* (or FM transform) and \mathcal{E} is its kernel. See [\[Huy06\]](#) for a thorough introduction to FM transforms. For example, if $\Delta: Z \rightarrow Z \times Z$ is the diagonal map and \mathcal{L} is in $\mathrm{Pic}(Z)$, then $\mathrm{FM}_{\Delta_* \mathcal{L}}(F) = F \otimes \mathcal{L}$. In particular, $\mathrm{FM}_{\mathcal{O}_{\Delta}}$ is the identity functor.

Convention. We will write $M_{\mathcal{L}}$ for the functor $\mathrm{FM}_{\Delta_* \mathcal{L}}$.

The calculus of FM transforms is, of course, not restricted to smooth and projective varieties. Note that f_* maps $\mathrm{D}^b(Z)$ to $\mathrm{D}^b(Z')$ as soon as $f: Z \rightarrow Z'$ is proper. In order to be able to control the tensor product and pullbacks, one can restrict to perfect complexes. Recall that a complex of sheaves on a quasi-projective variety Z is called *perfect* if it is locally quasi-isomorphic to a bounded complex of locally free sheaves. The triangulated category of perfect complexes on Z is denoted by $\mathrm{D}^{\mathrm{perf}}(Z)$. It is a full subcategory of $\mathrm{D}^b(Z)$. These two categories coincide if and only if Z is smooth.

We will sometimes take cones of morphisms between FM transforms. Of course, one needs to make sure that these cones actually exist. Luckily, if one works with FM transforms, this is not a problem, because the maps between the functors come from the underlying kernels

and everything works out, even for (reasonable) schemes which are not necessarily smooth and projective; see [AL12].

2.2. Group actions and derived categories. Let G be a finite group acting on a smooth variety X . Recall that a G -equivariant coherent sheaf is a pair (F, λ_g) , where $F \in \text{Coh}(X)$ and $\lambda_g: F \xrightarrow{\sim} g^*F$ are isomorphisms satisfying a cocycle condition. The category of G -equivariant coherent sheaves on X is denoted by $\text{Coh}^G(X)$. It is an abelian category. The *equivariant derived category*, denoted by $D_G^b(X)$, is defined as $D^b(\text{Coh}^G(X))$, see, for example, [Plo07] for details. Recall that for every subgroup $G' \subset G$ the restriction functor $\text{Res}: D_G^b(X) \rightarrow D_{G'}^b(X)$ has the induction functor $\text{Ind}: D_{G'}^b(X) \rightarrow D_G^b(X)$ as a left and right adjoint (see e.g. [Plo07, Sect. 1.4]). It is given for $F \in D^b(Z)$ by

$$(2) \quad \text{Ind}(F) = \bigoplus_{[g] \in G' \backslash G} g^* F$$

with the G -linearisation given by the G' -linearisation of F together with appropriate permutations of the summands.

If G acts trivially on X , there is also the functor $\text{triv}: D^b(X) \rightarrow D_G^b(X)$ which equips an object with the trivial G -linearisation. Its left and right adjoint is the functor $(_)^G: D_G^b(X) \rightarrow D^b(X)$ of invariants.

Given an equivariant morphism $f: X \rightarrow X'$ between varieties endowed with G -actions, there are equivariant pushforward and pullback functors, see, for example, [Plo07, Sect. 1.3] for details. We will sometimes write f_*^G for $(_)^G \circ f_*$. It is also well-known that the category $D_G^b(X)$ has a tensor product and the usual formulas, e.g. the adjunction formula, hold in the equivariant setting.

Finally, we need to recall that a character κ of G acts on the equivariant category by twisting the linearisation isomorphisms with κ . If $F \in D_G^b(X)$, we will write $F \otimes \kappa$ for this operation. We will tacitly use that twisting by characters commutes with the equivariant pushforward and pullback functors along G -equivariant maps.

2.3. Semi-orthogonal decompositions. References for the following facts are, for example, [Bon89] and [BO95].

Let \mathcal{T} be a Hom-finite triangulated category. A *semi-orthogonal decomposition* of \mathcal{T} is a sequence of full triangulated subcategories $\mathcal{A}_1, \dots, \mathcal{A}_m$ such that (a) if $A_i \in \mathcal{A}_i$ and $A_j \in \mathcal{A}_j$, then $\text{Hom}(A_i, A_j[l]) = 0$ for $i > j$ and all l , and (b) the \mathcal{A}_i generate \mathcal{T} , that is, the smallest triangulated subcategory of \mathcal{T} containing all the \mathcal{A}_i is already \mathcal{T} . We write $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$. If $m = 2$, these conditions boil down to the existence of a functorial exact triangle $A_2 \rightarrow T \rightarrow A_1$ for any object $T \in \mathcal{T}$.

A subcategory \mathcal{A} of \mathcal{T} is *right admissible* if the embedding functor ι has a right adjoint ι^R , *left admissible* if ι has a left adjoint ι^L , and *admissible* if it is left and right admissible.

Given any triangulated subcategory \mathcal{A} of \mathcal{T} , the full subcategory $\mathcal{A}^\perp \subseteq \mathcal{T}$ consists of objects T such that $\text{Hom}(A, T[k]) = 0$ for all $A \in \mathcal{A}$ and all $k \in \mathbb{Z}$. If \mathcal{A} is right admissible, then $\mathcal{T} = \langle \mathcal{A}^\perp, \mathcal{A} \rangle$ is a semi-orthogonal decomposition. Similarly, $\mathcal{T} = \langle \mathcal{A}, {}^\perp \mathcal{A} \rangle$ is a semi-orthogonal decomposition if \mathcal{A} is left admissible, where ${}^\perp \mathcal{A}$ is defined in the obvious way.

Examples typically arise from so-called exceptional objects. Recall that an object $E \in D^b(Z)$ (or any \mathbb{C} -linear triangulated category) is called *exceptional* if $\text{Hom}(E, E) = \mathbb{C}$ and $\text{Hom}(E, E[k]) = 0$ for all $k \neq 0$. The smallest triangulated subcategory containing E is then equivalent to $D^b(\text{Spec}(\mathbb{C}))$ and this category, by abuse of notation again denoted by E , is

admissible, leading to a semi-orthogonal decomposition $D^b(Z) = \langle E^\perp, E \rangle$. A sequence of objects E_1, \dots, E_n is called an *exceptional collection* if $D^b(Z) = \langle (E_1, \dots, E_n)^\perp, E_1, \dots, E_n \rangle$ and all E_i are exceptional. The collection is called *full* if $(E_1, \dots, E_n)^\perp = 0$.

Note that any fully faithful FM transform $\Phi: D^b(X) \rightarrow D^b(X')$ gives a semi-orthogonal decomposition $D^b(X') = \langle \Phi(D^b(X))^\perp, \Phi(D^b(X)) \rangle$, because any FM transform has a right and a left adjoint, see [Huy06, Prop. 5.9].

2.4. Dual semi-orthogonal decompositions. Let \mathcal{T} be a triangulated category together with a semi-orthogonal decomposition $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$. Then there is the *left-dual* semi-orthogonal decomposition $\mathcal{T} = \langle \mathcal{B}_n, \dots, \mathcal{B}_1 \rangle$ given by $\mathcal{B}_i := \langle \mathcal{A}_1, \dots, \mathcal{A}_{i-1}, \mathcal{A}_{i+1}, \dots, \mathcal{A}_n \rangle^\perp$. There is also a right-dual decomposition but we will always use the left-dual and refer to it simply as the *dual* semi-orthogonal decomposition. We summarise the properties of the dual semi-orthogonal decomposition needed later on in the following

Lemma 2.1. *Let $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ be a semi-orthogonal decomposition with dual semi-orthogonal decomposition $\mathcal{T} = \langle \mathcal{B}_n, \dots, \mathcal{B}_1 \rangle$.*

- (i) $\langle \mathcal{A}_1, \dots, \mathcal{A}_r \rangle = \langle \mathcal{B}_r, \dots, \mathcal{B}_1 \rangle$ and $\langle \mathcal{A}_1, \dots, \mathcal{A}_r \rangle^\perp = \langle \mathcal{B}_n, \dots, \mathcal{B}_{r+1} \rangle$ for $1 \leq r \leq n$.
- (ii) If $\langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ is given by an exceptional collection, i.e. $\mathcal{A}_i = \langle E_i \rangle$, then its dual is also given by an exceptional collection $\mathcal{B}_i = \langle F_i \rangle$ such that $\text{Hom}^*(E_i, F_j) = \delta_{ij} \mathbb{C}[0]$.

Proof. Part (i) is [Efi14, Prop. 2.7(i)]. Part (ii) is then clear. \square

An important classical example is the following

Lemma 2.2. *There are dual semi-orthogonal decompositions*

$$\begin{aligned} D^b(\mathbb{P}^{n-1}) &= \langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n-1) \rangle, \\ D^b(\mathbb{P}^{n-1}) &= \langle \Omega^{n-1}(n-1)[n-1], \dots, \Omega^1(1)[1], \mathcal{O} \rangle. \end{aligned}$$

Proof. The fact that both sequences are indeed full goes back to Beilinson, see [Huy06, Sect. 8.3] for an account. The fact that they are dual is classical and follows by a direct computation, for instance using [BS10, Lem. 2.5]. \square

The following relative version is the example of dual semi-orthogonal decompositions which we will need throughout the text.

Lemma 2.3. *Let $\nu: Z \rightarrow S$ be a \mathbb{P}^{n-1} -bundle. There is the semi-orthogonal decomposition*

$$D^b(Z) = \langle \nu^* D^b(S), \nu^* D^b(S) \otimes \mathcal{O}_\nu(1), \dots, \nu^* D^b(S) \otimes \mathcal{O}_\nu(n-1) \rangle$$

whose dual decomposition is given by

$$D^b(Z) = \langle \nu^* D^b(S) \otimes \Omega_\nu^{n-1}(n-1), \dots, \nu^* D^b(S) \otimes \Omega_\nu^1(1), \nu^* D^b(S) \rangle.$$

Proof. Part (i) is [Orl93, Thm. 2.6]. Part (ii) follows from Lemma 2.2. \square

The following consequence will be used in Subsection 4.5.

Corollary 2.4. *If $m < n$, there is the equality of subcategories of $D^b(Z)$*

$$\begin{aligned} &\langle \nu^* D^b(S) \otimes \mathcal{O}_\nu(m-n), \dots, \nu^* D^b(S) \otimes \mathcal{O}_\nu(-1) \rangle \\ &= \langle \nu^* D^b(S) \otimes \Omega_\nu^{n-1}(n-1), \dots, \nu^* D^b(S) \otimes \Omega_\nu^m(m) \rangle. \end{aligned}$$

Proof. Applying [Lemma 2.1\(i\)](#) to the dual decompositions of [Lemma 2.3](#) gives the equalities

$$\begin{aligned} & \langle \nu^* D^b(S) \otimes \Omega_\nu^{n-1}(n-1), \dots, \nu^* D^b(S) \otimes \Omega_\nu^m(m) \rangle \\ &= \langle \nu^* D^b(S), \dots, \nu^* D^b(S) \otimes \mathcal{O}_\nu(m-1) \rangle^\perp \\ &= \langle \nu^* D^b(S) \otimes \mathcal{O}_\nu(m-n), \dots, \nu^* D^b(S) \otimes \mathcal{O}_\nu(-1) \rangle. \end{aligned} \quad \square$$

2.5. Linear functors and linear semi-orthogonal decompositions. Let \mathcal{T} be a tensor triangulated category, i.e. a triangulated category with a compatible symmetric monoidal structure. Moreover, let \mathcal{X} be a *triangulated module category over \mathcal{T}* , i.e. there is an exact functor $\pi^*: \mathcal{T} \rightarrow \mathcal{X}$ and a tensor product $\otimes: \mathcal{T} \times \mathcal{X} \rightarrow \mathcal{X}$, that is an assignment $\pi^*(A) \otimes E$ functorial in $A \in \mathcal{T}$ and $E \in \mathcal{X}$.

We will take $\mathcal{T} = D_G^{\text{perf}}(Y)$ for some variety Y with an action by a finite group G . Note that $D_G^{\text{perf}}(Y)$ has a (derived) tensor product, and it is compatible with G -linearisations.

For \mathcal{X} , we have several cases in mind. If X is a smooth G -variety X with a G -equivariant morphism $\pi: X \rightarrow Y$, then we take $\mathcal{X} = D_G^b(X) = D_G^{\text{perf}}(X)$; this is a tensor triangulated category itself and π^* preserves these structures.

If Λ is an \mathcal{O}_Y -algebra, then let $\mathcal{X} = D^b(\Lambda)$ be the bounded derived category of finitely generated right Λ -modules with $\pi^*(A) = A \otimes_{\mathcal{O}_Y} \Lambda$ and $\pi^*(A) \otimes E = \pi^*(A) \otimes_{\mathcal{O}_Y} \Lambda \otimes_\Lambda E = \pi^*(A) \otimes_{\mathcal{O}_Y} E \in \mathcal{X}$. Note that if Λ is not commutative, then \mathcal{X} is not a tensor category.

We say that a full triangulated subcategory $\mathcal{A} \subset \mathcal{X}$ is \mathcal{T} -linear (since in our cases we have $\mathcal{T} = D^{\text{perf}}(Y)$ we will also speak of Y -linearity) if

$$\pi^*(A) \otimes E \in \mathcal{A} \quad \text{for all } A \in \mathcal{T} \text{ and } E \in \mathcal{X}.$$

We say that a semi-orthogonal decomposition $\mathcal{X} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ is \mathcal{T} -linear, if all the \mathcal{A}_i are \mathcal{T} -linear subcategories.

We call a class of objects $\mathcal{S} \subset \mathcal{X}$ (left/right) *spanning over \mathcal{T}* if $\pi^* \mathcal{T} \otimes \mathcal{S}$ is a (left/right) spanning class of \mathcal{X} in the non-relative sense. Recall that a subset $\mathcal{C} \subset \mathcal{X}$ is *generating* if $\mathcal{X} = \langle \mathcal{C} \rangle$ is the smallest triangulated category closed under direct summands containing \mathcal{C} . The subset $\mathcal{C} \subset \mathcal{X}$ is called *generating over \mathcal{T}* if $\mathcal{C} \otimes \pi^* \mathcal{T}$ generates $D^b(\mathcal{X})$.

Let \mathcal{X}' be a further tensor triangulated category together with a tensor triangulated functor $\pi'^*: \mathcal{T} \rightarrow \mathcal{X}'$. We say that an exact functor $\Phi: \mathcal{X} \rightarrow \mathcal{X}'$ is \mathcal{T} -linear if there are functorial isomorphisms

$$\Phi(\pi^*(A) \otimes E) \cong \pi'^*(A) \otimes \Phi(E) \quad \text{for all } A \in \mathcal{T} \text{ and } E \in \mathcal{X}.$$

The verification of the following lemma is straight-forward.

Lemma 2.5.

- (i) If $\Phi: \mathcal{X} \rightarrow \mathcal{X}'$ is \mathcal{T} -linear, then $\Phi(\mathcal{X})$ is a \mathcal{T} -linear subcategory of \mathcal{X}' .
- (ii) Let $\mathcal{A} \subset D^b(\mathcal{Y})$ be a \mathcal{T} -linear (left/right) admissible subcategory. Then the essential image of \mathcal{A} is $D^b(\mathcal{Y})$ if and only if \mathcal{A} contains a (left/right) spanning class over \mathcal{T} .

For the following, we consider the case that $\mathcal{X} = D^b(X)$ for some smooth variety X together with a proper morphism $\pi: X \rightarrow Y$.

Lemma 2.6. Let $\mathcal{A}, \mathcal{B} \subset D^b(\mathcal{X})$ be Y -linear full subcategories. Then

$$\mathcal{A} \subset \mathcal{B}^\perp \iff \pi_* \mathcal{H}om(B, A) = 0 \quad \forall A \in \mathcal{A}, B \in \mathcal{B}.$$

Proof. The direction \Leftarrow follows immediately from $\mathrm{Hom}^*(B, A) \cong \Gamma(Y, \pi_* \mathcal{H}om(B, A))$; recall that all our functors are the derived versions.

Conversely, assume that there are $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $\pi_* \mathcal{H}om(B, A) \neq 0$. Since $\mathrm{D}^{\mathrm{perf}}(Y)$ spans $\mathrm{D}(\mathrm{QCoh}(Y))$, this implies that there is an $E \in \mathrm{D}^{\mathrm{perf}}(Y)$ such that

$$\begin{aligned} 0 \neq \mathrm{Hom}^*(E, \pi_* \mathcal{H}om(B, A)) &\cong \Gamma(Y, \pi_* \mathcal{H}om(B, A) \otimes E^\vee) \cong \Gamma(Y, \pi_*(\mathcal{H}om(B, A) \otimes \pi^* E^\vee)) \\ &\cong \Gamma(Y, \pi_* \mathcal{H}om(B \otimes \pi^* E, A)) \\ &\cong \mathrm{Hom}^*(B \otimes \pi^* E, A). \end{aligned}$$

By the Y -linearity, we have $B \otimes \pi^* E \in \mathcal{B}$ and hence $\mathcal{A} \not\subset \mathcal{B}^\perp$. \square

2.6. Relative Fourier–Mukai transforms. Let $\pi: X \rightarrow Y$ and $\pi': X' \rightarrow Y$ be proper morphisms of varieties with X and X' being smooth. We denote the closed embedding of the fibre product into the product by $i: X \times_Y X' \hookrightarrow X \times X'$.

We call $\Phi: \mathrm{D}^b(X) \rightarrow \mathrm{D}^b(X')$ a *relative FM transform* if $\Phi = \mathrm{FM}_{i_* \mathcal{P}}$ for some object $\mathcal{P} \in \mathrm{D}^b(X \times_Y X')$. It is a standard computation that a relative FM transform is linear over Y , with respect to the pullbacks π^* and π'^* . Furthermore, we have $\Phi \cong p_*(q^*(_) \otimes \mathcal{P})$ where p and q are the projections of the fibre diagram

$$(3) \quad \begin{array}{ccc} & X \times_Y X' & \\ q \swarrow & & \searrow p \\ X & & X' \\ \pi \searrow & & \swarrow \pi' \\ & Y & \end{array}$$

The right adjoint of Φ is given by $\Phi^R := q_*(p^!(_) \otimes \mathcal{P}^\vee): \mathrm{D}^b(X') \rightarrow \mathrm{D}^b(X)$. We also have the following slightly stronger statement which one could call *relative adjointness*.

Lemma 2.7. *For $E \in \mathrm{D}^b(X)$ and $F \in \mathrm{D}^b(X')$, there are functorial isomorphisms*

$$\pi'_* \mathcal{H}om(\Phi(E), F) \cong \pi_* \mathcal{H}om(E, \Phi^R(F)).$$

Proof. This follows by Grothendieck duality, commutativity of (3), and projection formula:

$$\begin{aligned} \pi'_* \mathcal{H}om(\Phi(E), F) &\cong \pi'_* \mathcal{H}om(p_*(q^* E \otimes \mathcal{P}), F) \cong \pi'_* p_* \mathcal{H}om(q^* E \otimes \mathcal{P}, p^! F) \\ &\cong \pi_* q_* \mathcal{H}om(q^* E, p^! F \otimes \mathcal{P}^\vee) \\ &\cong \pi_* \mathcal{H}om(E, q_*(p^! F \otimes \mathcal{P}^\vee)) \\ &\cong \pi_* \mathcal{H}om(E, \Phi^R(F)). \end{aligned} \quad \square$$

For $E, F \in \mathrm{D}^b(X)$, using the isomorphism of the previous lemma, we can construct a natural morphism $\tilde{\Phi}: \pi_* \mathcal{H}om(E, F) \rightarrow \pi'_* \mathcal{H}om(\Phi(E), \Phi(F))$ as the composition

$$(4) \quad \tilde{\Phi}: \pi_* \mathcal{H}om(E, F) \rightarrow \pi'_* \mathcal{H}om(E, \Phi^R \Phi(F)) \cong \pi'_* \mathcal{H}om(\Phi(E), \Phi(F))$$

where the first morphism is induced by the unit of adjunction $F \rightarrow \Phi^R \Phi(F)$. Note that taking global sections gives back the functor Φ on morphisms, i.e. $\Phi = \Gamma(Y, \tilde{\Phi})$ as maps

$$\mathrm{Hom}^*(E, F) \cong \Gamma(Y, \pi_* \mathcal{H}om(E, F)) \rightarrow \Gamma(Y, \pi'_* \mathcal{H}om(\Phi(E), \Phi(F))) \cong \mathrm{Hom}^*(\Phi(E), \Phi(F)).$$

More generally, Φ induces functors for open subsets $U \subseteq Y$,

$$\Phi_U: D^b(W) \rightarrow D^b(W'), \quad \text{where } W = \pi^{-1}(U) \subseteq X \text{ and } W' = \pi'^{-1}(U) \subseteq X',$$

given by restricting the FM kernel $\iota_* \mathcal{P}$ to $W \times W'$ and we have $\Phi_U = \Gamma(U, \tilde{\Phi})$. From this we see that $\tilde{\Phi}$ is compatible with composition which means that the following diagram, for $E, F, G \in D^b(X)$, commutes

$$(5) \quad \begin{array}{ccc} \pi_* \mathcal{H}om(F, G) \otimes \pi_* \mathcal{H}om(E, F) & \xrightarrow{\quad} & \pi_* \mathcal{H}om(E, G) \\ \tilde{\Phi} \otimes \tilde{\Phi} \downarrow & & \downarrow \tilde{\Phi} \\ \pi'_* \mathcal{H}om(\Phi(F), \Phi(G)) \otimes \pi'_* \mathcal{H}om(\Phi(E), \Phi(F)) & \xrightarrow{\quad} & \pi'_* \mathcal{H}om(\Phi(E), \Phi(G)). \end{array}$$

2.7. Relative tilting bundles. Let $\pi: X \rightarrow Y$ be a proper morphism of varieties and let X be smooth. We say that $V \in D^b(X)$ is a *relative tilting object* if $\Lambda_V := \Lambda := \pi_* \mathcal{H}om(V, V)$ is cohomologically concentrated in degree 0 and V is a spanning class over Y . Note that Λ is an \mathcal{O}_Y -algebra. We denote the bounded derived category of coherent right modules over Λ by $D^b(\Lambda)$. It is a triangulated module category over $D^{\text{perf}}(Y)$ via $\pi^* A = A \otimes_{\mathcal{O}_Y} \Lambda$, and Λ is a relative generator. In particular, for $A \in D^b(X)$ and $M \in D^b(\Lambda)$, the tensor product $A \otimes M$ is over the base \mathcal{O}_Y . We get a relative tilting equivalence:

Proposition 2.8. *Let $V \in D^b(X)$ be a relative tilting object over Y . Then V generates $D^b(X)$ over Y , and the following functor defines a Y -linear exact equivalence:*

$$t_V := \pi_* \mathcal{H}om(V, _): D^b(X) \xrightarrow{\sim} D^b(\Lambda).$$

Proof. The Y -linearity of t_V is due to the projection formula

$$t_V(\pi^* A \otimes E) = \pi_*(\pi^* A \otimes \mathcal{H}om(V, E)) \cong A \otimes \pi_* \mathcal{H}om(V, E) = A \otimes t_V(E).$$

Consider the restricted functor $t'_V: \mathcal{V} := \langle V \otimes \pi^* D^{\text{perf}}(Y) \rangle \rightarrow D^b(\Lambda)$. We show that t'_V is fully faithful, using the adjunctions $\pi^* \dashv \pi_*$ and $_ \otimes_{\mathcal{O}_Y} \Lambda \dashv \text{For}$ where $\text{For}: D^b(\Lambda) \rightarrow D^b(Y)$ is scalar restriction, the projection formula, and the Y -linearity of t'_V :

$$\begin{aligned} \text{Hom}_{\mathcal{O}_X}(\pi^* A \otimes V, \pi^* B \otimes V) &\cong \text{Hom}_{\mathcal{O}_X}(\pi^* A, \pi^* B \otimes \mathcal{H}om(V, V)) \\ &\cong \text{Hom}_{\mathcal{O}_Y}(A, \pi_*(\pi^* B \otimes \mathcal{H}om(V, V))) \\ &\cong \text{Hom}_{\mathcal{O}_Y}(A, B \otimes \Lambda) \\ &\cong \text{Hom}_{\Lambda}(A \otimes \Lambda, B \otimes \Lambda) \\ &\cong \text{Hom}_{\Lambda}(t'_V(\pi^* A \otimes V), t'_V(\pi^* B \otimes V)) \end{aligned}$$

Since objects of type $\pi^* A \otimes V$ generate \mathcal{V} , this shows that t'_V is fully faithful. We have $t'_V(V) = \Lambda$. Since Λ is a relative generator, hence a relative spanning class, of $D^b(\Lambda)$, we get an equivalence $\mathcal{V} \cong D^b(\Lambda)$; see [Lemma 2.5](#).

We now claim that the inclusion $\mathcal{V} \hookrightarrow D^b(X)$ has a right adjoint, namely

$$t_V^{-1} t_V: D^b(X) \rightarrow D^b(\Lambda) \rightarrow \mathcal{V}.$$

For this, take $A \in D^{\text{perf}}(Y)$, $F \in D^b(X)$ and compute

$$\begin{aligned} \text{Hom}_{\mathcal{O}_X}(\pi^* A \otimes V, F) &\cong \text{Hom}_{\mathcal{O}_Y}(A, \pi_* \mathcal{H}om(V, F)) \cong \text{Hom}_{\Lambda}(A \otimes \Lambda, t_V(F)) \\ &\cong \text{Hom}_{\mathcal{V}}(t_V^{-1}(A \otimes \Lambda), t_V^{-1} t_V(F)) \\ &\cong \text{Hom}_{\mathcal{V}}(\pi^* A \otimes V, t_V^{-1} t_V(F)) \end{aligned}$$

where we use the projection formula, the adjunction $\Lambda \otimes_{\mathcal{O}_Y} - \dashv \text{For}$, the fact that $t_V'^{-1}$ is an equivalence, hence fully faithful, and the Y -linearity of $t_V'^{-1}$.

Since the right-admissible Y -linear subcategory $\mathcal{V} \subset \text{D}^b(X)$ contains the relative spanning class V , we get $\mathcal{V} = \text{D}^b(X)$ by [Lemma 2.5](#). This shows that V is a relative generator and that $t_V = t_V'$ is an equivalence. \square

Let $\pi': X' \rightarrow Y$ be a second proper morphism and let $\Phi: \text{D}^b(X) \xrightarrow{\sim} \text{D}^b(X')$ be a relative FM transform.

Lemma 2.9. *If*

$$\tilde{\Phi}_\Lambda := \tilde{\Phi}(V, V): \Lambda_V = \pi_* \mathcal{H}om(V, V) \rightarrow \pi'_* \mathcal{H}om(\Phi(V), \Phi(V)) = \Lambda_{\Phi(V)}$$

is an isomorphism, then the following diagram of functors commutes:

$$(6) \quad \begin{array}{ccc} \text{D}^b(X) & \xrightarrow{t_V} & \text{D}^b(\Lambda_V) \\ \Phi \downarrow & & \downarrow -\otimes_{\Lambda_V} \Lambda_{\Phi(V)} \\ \text{D}^b(X') & \xrightarrow{t_{\Phi(V)}} & \text{D}^b(\Lambda_{\Phi(V)}) \end{array}$$

Proof. We first show that $\tilde{\Phi}(V, E): t_V(E) \rightarrow t_{\Phi(V)}(\Phi(E))$ is an isomorphism in $\text{D}^b(Y)$ for every $E \in \text{D}(X)$. Assume first that there is an exact triangle $\pi^*A \otimes V \rightarrow E \rightarrow \pi^*B \otimes V$ for some $A, B \in \text{D}^{\text{perf}}(Y)$ and consider the induced morphism of triangles

$$\begin{array}{ccccc} \pi_* \mathcal{H}om(V, \pi^*A \otimes V) & \longrightarrow & \pi_* \mathcal{H}om(V, E) & \longrightarrow & \pi_* \mathcal{H}om(V, \pi^*B \otimes V) \\ \downarrow \tilde{\Phi}(V, \pi^*A \otimes V) & & \downarrow \tilde{\Phi}(V, E) & & \downarrow \tilde{\Phi}(V, \pi^*B \otimes V) \\ \pi'_* \mathcal{H}om(\Phi(V), \Phi(\pi^*A \otimes V)) & \longrightarrow & \pi'_* \mathcal{H}om(\Phi(V), \Phi(E)) & \longrightarrow & \pi'_* \mathcal{H}om(\Phi(V), \Phi(\pi^*B \otimes V)). \end{array}$$

The outer vertical arrows are isomorphisms because they decompose as

$$\begin{aligned} \pi_* \mathcal{H}om(V, V \otimes \pi^*A) &\xrightarrow{\sim} \pi_* \mathcal{H}om(V, V) \otimes A \xrightarrow[\tilde{\Phi}_\Lambda]{\sim} \pi'_* \mathcal{H}om(\Phi(V), \Phi(V)) \otimes A \\ &\xrightarrow{\sim} \pi'_* \mathcal{H}om(\Phi(V), \Phi(V) \otimes \pi'^*A) \xrightarrow{\sim} \pi'_* \mathcal{H}om(\Phi(V), \Phi(V \otimes \pi^*A)). \end{aligned}$$

Therefore, the middle vertical arrow is an isomorphism as well. Since V is a relative generator, we can show that $\tilde{\Phi}(V, E)$ is an isomorphism for arbitrary $E \in \text{D}^b(X)$ by repeating the above argument.

Using the commutativity of (5) with $E = F = G = V$, we see that $\tilde{\Phi}(V, E)$ induces an $\Lambda_{\Phi(V)}$ -linear isomorphism $\pi_* \mathcal{H}om(V, E) \otimes_{\Lambda_V} \Lambda_{\Phi(V)} \xrightarrow{\sim} \pi'_* \mathcal{H}om(\Phi(V), \Phi(E))$. \square

Lemma 2.10. *The functor Φ is fully faithful if and only if $\tilde{\Phi}_\Lambda: \Lambda_V \rightarrow \Lambda_{\Phi(V)}$ is an isomorphism.*

Proof. If Φ is fully faithful, the unit $\text{id} \rightarrow \Phi^R \Phi$ is an isomorphism. Hence, $\tilde{\Phi}_\Lambda$ is an isomorphism; see (4).

Conversely, let $\tilde{\Phi}_\Lambda$ be an equivalence. By Lemma 2.9, we get a commutative diagram

$$\begin{array}{ccc} D^b(X) & \xrightarrow{t_V} & D^b(\Lambda_V) \\ \Phi \downarrow & & \downarrow -\otimes_{\Lambda_V} \Lambda_{\Phi(V)} \\ \langle \Phi(V) \rangle & \xrightarrow{t_{\Phi(V)}} & D^b(\Lambda_{\Phi(V)}). \end{array}$$

In this diagram, the horizontal functors are tilting equivalences. The right-hand vertical functor is an equivalence, too, by assumption on $\tilde{\Phi}_\Lambda$. Hence, $\Phi: D^b(X) \rightarrow \langle \Phi(V) \rangle$ is an equivalence, which implies that $\Phi: D^b(X) \rightarrow D^b(X')$ is fully faithful. \square

Lemma 2.11. *Let $V \in D^b(X)$ be a relative tilting object, $\Phi: D^b(X) \xrightarrow{\sim} D^b(X)$ a relative FM autoequivalence, and $\nu: V \xrightarrow{\sim} \Phi(V)$ an isomorphism such that*

$$\tilde{\Phi}_\Lambda = \nu \circ _ \circ \nu^{-1}: \pi_* \mathcal{H}om(V, V) \rightarrow \pi_* \mathcal{H}om(\Phi(V), \Phi(V)),$$

i.e. $\Phi_U(\varphi) \circ \nu = \nu \circ \varphi$ for all open subsets $U \subset Y$ and $\varphi \in \Lambda_V(U)$. Then there exists an isomorphism of functors $\text{id} \xrightarrow{\sim} \Phi$ restricting to ν .

Proof. We claim that, under our assumptions, the following diagram of functors commutes

$$(7) \quad \begin{array}{ccc} D^b(X) & \xrightarrow{t_V} & D^b(\Lambda_V) \\ \text{id} \downarrow & & \downarrow -\otimes_{\Lambda_V} \Lambda_{\Phi(V)} \\ D^b(X) & \xrightarrow{t_{\Phi(V)}} & D^b(\Lambda_{\Phi(V)}) \end{array}$$

We construct a natural isomorphism $\eta: t_{\Phi V} \xrightarrow{\sim} t_V \otimes_{\Lambda_V} \Lambda_{\Phi V}$ as follows. For $E \in D^b(X)$, there is a natural \mathcal{O}_Y -linear isomorphism $\pi_* \mathcal{H}om(\Phi(V), E) \xrightarrow{\sim} \pi_* \mathcal{H}om(V, E) \otimes_{\Lambda_V} \Lambda_{\Phi(V)}$ given by $f \mapsto f\nu \otimes 1$; the inverse map is $g \otimes 1 \mapsto g\nu^{-1}$. This map is linear over $\Lambda_{\Phi V}$ because, for a local section $\lambda \in \pi_* \mathcal{H}om(\Phi(V), \Phi(V))$, we have by our assumption, setting $\varphi = \tilde{\Phi}^{-1}(\lambda)$:

$$\eta(f\lambda) = f\lambda\nu \otimes 1 = f\nu\tilde{\Phi}^{-1}(\lambda) \otimes 1 = f\nu \otimes \lambda = (f\nu \otimes 1)\lambda.$$

Comparing the diagrams (7) and (6) shows that $\Phi \cong \text{id}$. \square

Corollary 2.12. *Let $V \in D^b(X)$ be a relative tilting object, $\Phi_1, \Phi_2: D^b(X) \xrightarrow{\sim} D^b(X')$ relative FM equivalences, and $\nu: \Phi_1(V) \xrightarrow{\sim} \Phi_2(V)$ an isomorphism such that $\Phi_{2,U}(\varphi) \circ \nu = \nu \circ \Phi_{1,U}(\varphi)$ for all $\varphi \in \Lambda_V(U)$ and $U \subset Y$ open. Then there exists a isomorphism of functors $\Phi_1 \xrightarrow{\sim} \Phi_2$ restricting to ν .*

Moreover, if $V = L_1 \oplus \cdots \oplus L_k$ decomposes as a direct sum, then the above condition is satisfied by specifying isomorphisms $\nu_i: \Phi_1(L_i) \xrightarrow{\sim} \Phi_2(L_i)$ inducing functor isomorphisms $\Phi_{1,U} \xrightarrow{\sim} \Phi_{2,U}$ on the full finite subcategory $\{L_{1|U}, \dots, L_{k|U}\}$ of $D^b(\pi^{-1}(U))$.

Remark 2.13. All the results of this subsection remain valid in an equivariant setting, where a finite group G acts on X and $\pi: X \rightarrow Y$ is G -invariant. Then the correct sheaf of \mathcal{O}_Y -algebras is $\Lambda_V = \pi_*^G \mathcal{H}om(V, V)$.

2.8. Spherical functors. An exact functor $\varphi: \mathcal{C} \rightarrow \mathcal{D}$ between triangulated categories is called *spherical* if it admits both adjoints, if the cone endofunctor $F[1] := \text{cone}(\text{id}_{\mathcal{C}} \rightarrow \varphi^R \varphi)$ is an autoequivalence of \mathcal{C} , and if the canonical functor morphism $\varphi^R \rightarrow F\varphi^L[1]$ is an isomorphism. A spherical functor is called *split* if the triangle defining F is split. The proper framework for dealing with functorial cones are dg-categories; the triangulated categories in this article are of geometric nature, and we can use Fourier–Mukai transforms. See [AL13] for proofs in great generality.

Given a spherical functor $\varphi: \mathcal{C} \rightarrow \mathcal{D}$, the cone of the natural transformation $\mathsf{T} = \mathsf{T}_{\varphi} := \text{cone}(\varphi\varphi^R \rightarrow \text{id}_{\mathcal{D}})$ is called the *twist around* φ ; it is an autoequivalence of \mathcal{D} .

The following lemma follows immediately from the definition, since an equivalence has its inverse functor as both left and right adjoint.

Lemma 2.14. *Let $\varphi: \mathcal{C} \rightarrow \mathcal{D}$ be a spherical functor and let $\delta: \mathcal{D} \rightarrow \mathcal{D}'$ be an equivalence. Then $\delta \circ \varphi: \mathcal{C} \rightarrow \mathcal{D}'$ is a spherical functor with associated twist functor $\mathsf{T}_{\delta\varphi} = \delta \mathsf{T}_{\varphi} \delta^{-1}$.*

3. THE GEOMETRIC SETUP

Let X be a smooth quasi-projective variety together with an action of a finite group G . Let $S := \text{Fix}(G)$ be the locus of fixed points. Then $S \subset X$ is a closed subset, which is automatically smooth since, locally in the analytic topology, the action can be linearised by Cartan’s lemma, see [Car57, Lem. 2]. Also note that X/G has rational singularities, like any quotient singularity over \mathbb{C} [Kov00].

Condition 3.1. We make strong assumptions on the group action:

- (i) $G \cong \mu_m$ is a cyclic group. Fix a generator $g \in G$.
- (ii) Only the trivial isotropic groups 1 and μ_m occur.
- (iii) The generator g acts on the normal bundle $N := N_{S/X}$ by multiplication with some fixed primitive m -th root of unity ζ .

Condition (ii) obviously holds if m is prime.

Condition (iii) can be rephrased: there is a splitting $T_{X|S} = T_S \oplus N_{S/X}$ because T_S is the subsheaf of G -invariants of $T_{X|S}$ and we work over characteristic 0. By (iii), this is even the splitting into the eigenbundles corresponding to the eigenvalues 1 and ζ . We denote by $\chi: G \rightarrow \mathbb{C}^*$ the character with $\chi(g) = \zeta^{-1}$. Hence, we can reformulate (iii) by saying that G acts on N via χ^{-1} .

From these assumptions we deduce the following commutative diagram

$$(8) \quad \begin{array}{ccccc} \tilde{X} & \xrightarrow{p} & X \\ & \nwarrow j & \nearrow a \\ & Z = \mathbb{P}(N) & \xrightarrow{\nu} S \\ & \swarrow i & \searrow b \\ \tilde{X}/G = \tilde{Y} & \xrightarrow{q} & Y = X/G \end{array}$$

$\text{blow-up in } S$ (above p and q)
 $\text{blow-up in } S$ (below i and b)
 π (right of X and Y)
 q (left of \tilde{X} and \tilde{Y})

where a, b, i , and j are closed embeddings and π is the quotient morphism. The G -action on X lifts to a G -action on \tilde{X} . Since, by assumption, G acts diagonally on N , it acts trivially on the exceptional divisor $Z = \mathbb{P}(N)$. In particular, the fixed point locus of the G -action on \tilde{X} is

a divisor. Hence, the quotient variety \tilde{Y} is again smooth and the quotient morphism q is flat due to the Chevalley–Shephard–Todd theorem. Since the composition $\pi \circ p$ is G -invariant, it induces the morphism $\varrho: \tilde{Y} \rightarrow Y$ which is easily seen to be birational, hence a resolution of singularities. The preimage $\varrho^{-1}(S)$ of the singular locus is a divisor in \tilde{Y} . Hence, by the universal property of the blow-up, we get a morphism $\tilde{Y} \rightarrow \text{Bl}_S Y$ which is easily seen to be an isomorphism.

3.1. The resolution as a moduli space of G -clusters. The result of this section might be of independent interest. Let X be a smooth quasi-projective variety and G a finite group acting on X . A G -cluster on X is a closed zero-dimensional G -invariant subscheme $W \subset X$ such that the G -representation $H^0(W, \mathcal{O}_W)$ is isomorphic to the regular representation of G . There is a fine moduli space $\text{Hilb}^G(X)$ of G -clusters, called the G -Hilbert scheme. It is equipped with the *equivariant Hilbert–Chow morphism* $\tau: \text{Hilb}^G(X) \rightarrow X/G, W \mapsto \text{supp}(W)$, mapping G -clusters to their underlying G -orbits.

Proposition 3.2. *Let G be a finite cyclic group acting on X such that all isotropy groups are either 1 or G , and such that G acts on the normal bundle $N_{\text{Fix}(G)/X}$ by scalars which means that [Condition 3.1](#) is satisfied. Then there is an isomorphism*

$$\varphi: \tilde{Y} \xrightarrow{\cong} \text{Hilb}^G(X) \quad \text{with} \quad \tau \circ \varphi = \varrho.$$

Proof. We use the notation from (8). One can identify \tilde{X} with the reduced fibre product $(\tilde{Y} \times_Y X)_{\text{red}}$ which gives a canonical embedding $\tilde{X} \subset \tilde{Y} \times X$. Under this embedding, the generic fibre of q is a reduced free G -orbit of the action on X . In particular, it is a G -cluster. By the flatness of q , every fibre is a G -cluster and we get the classifying morphism $\varphi: \tilde{Y} \rightarrow \text{Hilb}^G(X)$ which is easily seen to satisfy $\tau \circ \varphi = \varrho$.

Let $s \in S$ and $z \in Z$ with $\nu(z) \in s$. Let $\ell \subset N(s)$ be the line corresponding to z . Then, one can check that the tangent space of the G -cluster $q^{-1}(i(z)) \subset X$ is exactly ℓ . Hence, the G -clusters in the family \tilde{X} are all different so that the classifying morphism φ is injective. For the bijectivity of φ , it is only left to show that the G -orbits supported on a given fixed point $s \in S$ are parametrised by $\mathbb{P}(N(s))$. Let $\xi \subset X$ be such a G -cluster. In particular, ξ is a length $m = |G|$ subscheme concentrated in s and hence can be identified with an ideal $I \subset \mathcal{O}_{X,s}/\mathfrak{m}_{X,s}^m$ of codimension m . By Cartan’s lemma, the G -action on X can be linearised in an analytic neighbourhood of s . Hence, there is an G -equivariant isomorphism

$$\mathcal{O}_{X,s}/\mathfrak{m}_{X,s}^m \cong \mathbb{C}[x_1, \dots, x_k, y_1, \dots, y_n]/(x_1, \dots, x_k, y_1, \dots, y_n)^m =: R$$

where G acts trivially on the x_i and by multiplication by ζ^{-1} on the y_i . Furthermore, $n = \text{rank } N_{S/X}$ and $k = \text{rank } T_S = \dim X - n$. By assumption, $\mathcal{O}(\xi)$ is the regular μ_m -representation. In other words,

$$(9) \quad \mathcal{O}(\xi) \cong R/I \cong \chi^0 \oplus \chi \oplus \dots \oplus \chi^{m-1}$$

where χ is the character given by multiplication by ζ^{-1} . In particular, R/I has a one-dimensional subspace of invariants. It follows that every x_i is congruent to a constant polynomial modulo I . Hence, we can make an identification $\mathcal{O}(\xi) \cong R'/J$ where J is a G -invariant ideal in $R' = \mathbb{C}[y_1, \dots, y_k]/\mathfrak{n}^m$ where $\mathfrak{n} = (y_1, \dots, y_n)$. The decomposition of the G -representation R' into eigenspaces is exactly the decomposition into the spaces of homogeneous polynomials. Hence, an ideal $J \subset R'$ is G -invariant if and only if it is homogeneous.

Furthermore, (9) implies that

$$\dim_{\mathbb{C}}(\mathfrak{n}^i / (J \cap \mathfrak{n}^i + \mathfrak{n}^{i+1})) = 1 \quad \text{for all } i = 0, \dots, m-1$$

which means that ξ is curvilinear. In summary ξ can be identified with a homogeneous curvilinear ideal J in R' . The choice of such a J corresponds to a point in $\mathbb{P}((\mathfrak{n}/\mathfrak{n}^2)^\vee) \cong \mathbb{P}(N(s))$; see [Göt94, Rem. 2.1.7].

Hence, φ is a bijection and we only need to show that $\text{Hilb}^G(X)$ is smooth. The smoothness in points representing free orbits is clear since the G -Hilbert–Chow morphism is an isomorphism on the locus of these points. So it is sufficient to show that

$$\text{Hom}_{\text{D}_G^b(X)}^1(\mathcal{O}_\xi, \mathcal{O}_\xi) = \dim X = n + k$$

for a G -cluster ξ supported on a fixed point. Following the above arguments, we have

$$\text{Hom}_{\text{D}_G^b(X)}^*(\mathcal{O}_\xi, \mathcal{O}_\xi) \cong \text{Hom}_{\text{D}_G^b(\mathbb{A}^k \times \mathbb{A}^n)}^*(\mathcal{O}_{\xi'}, \mathcal{O}_{\xi'})$$

where G acts trivially on \mathbb{A}^k and by multiplication by ζ on \mathbb{A}^n . Furthermore, by a transformation of coordinates, we may assume that

$$\xi' = V(x_1, \dots, x_k, y_1^m, y_2, \dots, y_n) \subset \mathbb{A}^k \times \mathbb{A}^n.$$

We have $\mathcal{O}_{\xi'} \cong \mathcal{O}_0 \boxtimes \mathcal{O}_\eta$ where

$$\eta = V(y_1^m, y_2, \dots, y_n) \subset \mathbb{A}^n.$$

By Künneth formula, we get

$$\begin{aligned} \text{Hom}_{\text{D}_G^b(\mathbb{A}^k \times \mathbb{A}^n)}^*(\mathcal{O}_{\xi'}, \mathcal{O}_{\xi'}) &\cong \text{Hom}_{\text{D}^b(\mathbb{A}^k)}^*(\mathcal{O}_0, \mathcal{O}_0) \otimes \text{Hom}_{\text{D}_G^b(\mathbb{A}^n)}^*(\mathcal{O}_\eta, \mathcal{O}_\eta) \\ &\cong \wedge^*(\mathbb{C}^k) \otimes \text{Hom}_{\text{D}_G^b(\mathbb{A}^n)}^*(\mathcal{O}_\eta, \mathcal{O}_\eta). \end{aligned}$$

Furthermore, $\text{Hom}_{\text{D}_G^b(\mathbb{A}^n)}^0(\mathcal{O}_\eta, \mathcal{O}_\eta) \cong H^0(\mathcal{O}_\eta)^G \cong \mathbb{C}$. Hence, it is sufficient to show that $\text{Hom}_{\text{D}_G^b(\mathbb{A}^n)}^1(\mathcal{O}_\eta, \mathcal{O}_\eta) \cong \mathbb{C}^n$. Note that η is contained in the line $\ell = V(y_2, \dots, y_n)$. On ℓ we have the Koszul resolution

$$0 \rightarrow \mathcal{O}_\ell \xrightarrow{\cdot y_1^m} \mathcal{O}_\ell \rightarrow \mathcal{O}_\eta \rightarrow 0.$$

Using this, we compute

$$\text{Hom}_{\text{D}^b(\ell)}^*(\mathcal{O}_\eta, \mathcal{O}_\eta) \cong \mathcal{O}_\eta[0] \oplus \mathcal{O}_\eta[-1].$$

Note that the normal bundle of ℓ , as an equivariant bundle, is given by $N_{\ell/\mathbb{A}^n} \cong (\mathcal{O}_\ell \otimes \chi^{-1})^{\oplus n-1}$. By [AC12, Thm. 1.4], we have

$$\begin{aligned} \text{Hom}_{\text{D}^b(\mathbb{A}^n)}^*(\mathcal{O}_\eta, \mathcal{O}_\eta) &\cong \text{Hom}_{\text{D}^b(\ell)}^*(\mathcal{O}_\eta, \mathcal{O}_\eta \otimes \wedge^* N_{\ell/\mathbb{A}^n}) \\ &\cong \text{Hom}_{\text{D}^b(\ell)}^*(\mathcal{O}_\eta, \mathcal{O}_\eta) \otimes \wedge^*((\mathcal{O}_\ell \otimes \chi^{-1})^{\oplus n-1}). \end{aligned}$$

Evaluating in degree 1 gives

$$\text{Hom}_{\text{D}^b(\mathbb{A}^n)}^1(\mathcal{O}_\eta, \mathcal{O}_\eta) \cong \mathcal{O}_\eta \oplus (\mathcal{O}_\eta \otimes \chi^{-1})^{\oplus n-1}.$$

Since, as a G -representation, $\mathcal{O}_\eta \cong \chi^0 \oplus \chi^1 \oplus \dots \oplus \chi^{m-1}$, we get an n -dimensional space of invariants

$$\text{Hom}_{\text{D}_G^b(\mathbb{A}^n)}^1(\mathcal{O}_\eta, \mathcal{O}_\eta) \cong \text{Hom}_{\text{D}^b(\mathbb{A}^n)}^1(\mathcal{O}_\eta, \mathcal{O}_\eta)^G \cong \mathbb{C}^n. \quad \square$$

The following lemma is needed later in [Subsection 4.4](#) but its proof fits better into this section.

Lemma 3.3. *Assume that $m = |G| \geq n = \text{codim}(S \hookrightarrow X)$. Let $\xi_1, \xi_2 \subset X$ be two different G -clusters supported on the same point $s \in S$. Then $\text{Hom}_{\text{D}_G^b(X)}^*(\mathcal{O}_{\xi_1}, \mathcal{O}_{\xi_2}) = 0$.*

Proof. By the same arguments as in the proof of the previous proposition we can reduce to the claim that

$$\text{Hom}_{\text{D}_G^b(\mathbb{A}^n)}^*(\mathcal{O}_{\eta_1}, \mathcal{O}_{\eta_2}) = 0$$

where $\eta_1 = V(y_1^m, y_2, \dots, y_n)$ and $\eta_2 = V(y_1, y_2^m, y_3, \dots, y_n)$. Set $\ell_1 = V(y_2, \dots, y_n)$, $\ell_2 = V(y_1, y_3, \dots, y_n)$, $E = \langle \ell_1, \ell_2 \rangle = V(y_3, \dots, y_n)$ and consider the diagram of closed embeddings

$$\begin{array}{ccccc} & & \ell_2 & & \\ & u \nearrow & \downarrow r & \nwarrow \iota_2 & \\ \{0\} & & E & \xrightarrow{t} & \mathbb{A}^n. \\ & v \searrow & \uparrow s & \nearrow \iota_1 & \\ & & \ell_1 & & \end{array}$$

where $N_t \cong (\mathcal{O}_E \otimes \chi^{-1})^{\oplus n-2}$. By [Kru14, Lem. 3.3] (alternatively, one may consult [Gri13] or [ACH14] for more general results on derived intersection theory), we get

$$\begin{aligned} \text{Hom}_{\text{D}^b(\mathbb{A}^n)}^*(\mathcal{O}_{\eta_1}, \mathcal{O}_{\eta_2}) &\cong \text{Hom}_{\text{D}^b(\mathbb{A}^n)}^*(\iota_{1*}\mathcal{O}_{\eta_1}, \iota_{2*}\mathcal{O}_{\eta_2}) \\ &\cong \text{Hom}_{\text{D}^b(\ell_2)}^*(\iota_2^*\iota_{1*}\mathcal{O}_{\eta_1}, \mathcal{O}_{\eta_2}) \\ &\cong \text{Hom}_{\text{D}^b(\ell_2)}^*(u_*v^*\mathcal{O}_{\eta_1}, \mathcal{O}_{\eta_2}) \otimes \wedge^* N_{t|\ell_2} \\ (10) \quad &\cong \text{Hom}_{\text{D}^b(\ell_2)}^*(\mathcal{O}_0, \mathcal{O}_{\eta_2}) \otimes \wedge^*(\mathcal{O}_{\ell_2} \otimes \chi^{-1})^{\oplus n-2}. \end{aligned}$$

We have the equivariant Koszul resolution

$$0 \rightarrow \mathcal{O}_\ell \otimes \chi \xrightarrow{y} \mathcal{O}_\ell \rightarrow 0$$

of \mathcal{O}_0 where we set for simplicity $\ell := \ell_2$ and $y := y_2$. Applying $\text{Hom}(_, \mathcal{O}_\eta)$ we get

$$0 \rightarrow \mathbb{C}[y]/y^m \otimes \xrightarrow{y} \mathbb{C}[y]/y^m \otimes \chi^{-1} \rightarrow 0.$$

Taking cohomology, we get

$$\text{Hom}_{\text{D}^b(\ell_2)}^*(\mathcal{O}_0, \mathcal{O}_{\eta_2}) \cong \mathbb{C}\langle y^{m-1} \rangle[0] \oplus \mathbb{C}\langle 1 \rangle \otimes \chi^{-1}[-1] \cong \mathcal{O}_0 \otimes \chi^{-1}[0] \oplus \mathcal{O}_0 \otimes \chi^{-1}[-1].$$

It follows that the graded vector space (10) does not have invariants (recall that $m \geq n$). \square

4. PROOF OF THE MAIN RESULT

In this section, we will study the derived categories $\text{D}^b(\tilde{Y})$ and $\text{D}_G^b(X)$ in the setup described in the previous section. In particular, we will prove Theorems 1.2 and 1.3.

We set $n = \text{codim}(S \hookrightarrow X)$ and $m = |G|$, in other words $G = \mu_m$. We consider, for $\alpha \in \mathbb{Z}/m\mathbb{Z}$ and $\beta \in \mathbb{Z}$, the exact functors

$$\begin{aligned} \Phi &:= p_* \circ q^* \circ \text{triv}: \text{D}^b(\tilde{Y}) \rightarrow \text{D}_G^b(X) \\ \Psi &:= (-)^G \circ q_* \circ p^*: \text{D}_G^b(X) \rightarrow \text{D}^b(\tilde{Y}) \\ \Theta_\beta &:= i_*(\nu^*(_) \otimes \mathcal{O}_\nu(\beta)): \text{D}^b(S) \rightarrow \text{D}^b(\tilde{Y}) \\ \Xi_\alpha &:= (a_* \circ \text{triv}) \otimes \chi^\alpha: \text{D}^b(S) \rightarrow \text{D}_G^b(X). \end{aligned}$$

With this notation, the precise version of [Theorem 1.2](#) is

Theorem 4.1.

- (i) *The functor Φ is fully faithful for $m \geq n$ and an equivalence for $m = n$. For $m > n$, all the functors Ξ_α are fully faithful and there is a semi-orthogonal decomposition*

$$\mathrm{D}_G^b(X) = \langle \Xi_{n-m}(\mathrm{D}^b(S)), \dots, \Xi_{-1}(\mathrm{D}^b(S)), \Phi(\mathrm{D}^b(\tilde{Y})) \rangle.$$

- (ii) *The functor Ψ is fully faithful for $n \geq m$ and an equivalence for $n = m$. For $n > m$, all the functors Θ_β are fully faithful and there is a semi-orthogonal decomposition*

$$\mathrm{D}^b(\tilde{Y}) = \langle \Theta_{m-n}(\mathrm{D}^b(S)), \dots, \Theta_{-1}(\mathrm{D}^b(S)), \Psi(\mathrm{D}_G^b(X)) \rangle.$$

Remark 4.2. We will see later in [Corollary 4.14](#) that $K_{\tilde{Y}} \leq \varrho^* K_Y$ for $m \geq n$ and $K_{\tilde{Y}} \geq \varrho^* K_Y$ for $n \geq m$. Hence, [Theorem 4.1](#) is in accordance with the DK-Hypothesis as described in the introduction.

For the proof, we first need some more preparations.

4.1. Generators and linearity.

Lemma 4.3. *The bundle $V := \mathcal{O}_X \otimes \mathbb{C}[G] = \mathcal{O}_X \otimes (\chi^0 \oplus \dots \oplus \chi^{m-1})$ is a relative tilting object for $\mathrm{D}_G^b(X)$ over $\mathrm{D}^{\mathrm{perf}}(Y)$.*

Proof. If $L \in \mathrm{Pic}(Y) \subset \mathrm{D}^{\mathrm{perf}}(Y)$ is an ample line bundle, then so is $\pi^*(L)$. Hence, $\mathrm{D}^b(X)$ has a generator of the form $E := \pi^*(\mathcal{O}_Y \oplus L \oplus \dots \oplus L^{\otimes k})$ for some $k \gg 0$; see [\[Orl09\]](#).

In particular, E is a spanning class of $\mathrm{D}^b(X)$. Using the adjunction $\mathrm{Res} \dashv \mathrm{Ind} \dashv \mathrm{Res}$, it follows that $\mathrm{Ind}(E) \cong E \oplus E \otimes \chi \oplus \dots \oplus E \otimes \chi^{m-1}$ is a spanning class of $\mathrm{D}_G^b(X)$. Hence, $V = \mathrm{Ind} \mathcal{O}_X$ is a relative spanning class of $\mathrm{D}_G^b(X)$ over $\mathrm{D}^{\mathrm{perf}}(Y)$.

Since V is a vector bundle, so is $\mathcal{H}om(V, V) = V^\vee \otimes V$. The map π is finite, hence π_* is exact (does not need to be derived). Finally, taking G -invariants is exact because we work in characteristic 0. Altogether, $\pi_*^G \mathcal{H}om(V, V)$ is a sheaf concentrated in degree 0. \square

Lemma 4.4. *The functors Φ and Ψ , and for all $\alpha, \beta \in \mathbb{Z}$ the subcategories*

$$\begin{aligned} \Xi_\alpha(\mathrm{D}^b(S)) &= a_*(\mathrm{D}^b(S)) \otimes \chi^\alpha \subset \mathrm{D}_G^b(X) \quad \text{and} \\ \Theta_\beta(\mathrm{D}^b(S)) &= i_* \nu^* \mathrm{D}^b(S) \otimes \mathcal{O}_{\tilde{Y}}(\beta) \subset \mathrm{D}^b(\tilde{Y}) \end{aligned}$$

are Y -linear for $\pi^ \mathrm{triv}: \mathrm{D}^{\mathrm{perf}}(Y) \rightarrow \mathrm{D}_G^b(X)$ and $\varrho^*: \mathrm{D}^{\mathrm{perf}}(Y) \rightarrow \mathrm{D}^b(\tilde{Y})$, respectively.*

Proof. We first show that Φ is Y -linear. Recall that in our setup this means

$$\Phi(\varrho^*(E) \otimes F) \cong \pi^* \mathrm{triv}(E) \otimes \Phi(F)$$

for any $E \in \mathrm{D}^{\mathrm{perf}}(Y)$ and $F \in \mathrm{D}^b(\tilde{Y})$. But this holds, since

$$\begin{aligned} \pi^* \mathrm{triv}(E) \otimes \Phi(F) &\cong \pi^* \mathrm{triv}(E) \otimes p_* q^* \mathrm{triv}(F) \\ &\cong p_*(p^* \pi^* \mathrm{triv}(E) \otimes q^* \mathrm{triv}(F)) \\ &\cong p_*(q^* \varrho^* \mathrm{triv}(E) \otimes q^* \mathrm{triv}(F)) \\ &\cong p_* q^* \mathrm{triv}(\varrho^*(E) \otimes F). \end{aligned}$$

The proof that Ψ is Y -linear is similar and is left to the reader.

The Y -linearity of the image categories follows from [Lemma 2.5\(i\)](#). \square

Lemma 4.5. *The set of sheaves $\mathcal{S} := \{\mathcal{O}_{\tilde{Y}}\} \cup \{i_{s*}\Omega^r(r) \mid s \in S, r = 0, \dots, n-1\}$ forms a spanning class of $D^b(\tilde{Y})$ over Y , where $i_s: \mathbb{P}^{n-1} \cong \varrho^{-1}(s) \hookrightarrow \tilde{Y}$ denotes the fibre embedding.*

Proof. We need to show that $\hat{\mathcal{S}} := \varrho^* D^{\text{perf}}(Y) \otimes \mathcal{S}$ is a spanning class of $D^b(\tilde{Y})$. Let $\tilde{y} \in \tilde{Y} \setminus Z$. Then $y = \varrho(\tilde{y})$ is a smooth point of Y . Hence, $\mathcal{O}_y \in D^{\text{perf}}(Y)$ and $\mathcal{O}_{\tilde{y}} \in \varrho^* D^{\text{perf}}(Y) = \varrho^* D^{\text{perf}}(Y) \otimes \mathcal{O}_{\tilde{Y}} \subset \hat{\mathcal{S}}$. Thus, an object $E \in D^b(\tilde{Y})$ with $\text{supp } E \cap (\tilde{Y} \setminus Z) \neq \emptyset$ satisfies $\text{Hom}^*(E, \hat{\mathcal{S}}) \neq 0 \neq \text{Hom}^*(\hat{\mathcal{S}}, E)$; see [Huy06, Lemma 3.29].

Let now $0 \neq E \in D^b(\tilde{Y})$ with $\text{supp } E \subset Z$. Then there exists $s \in S$ such that $i_s^* E \neq 0 \neq i_s^! E$; see again [Huy06, Lemma 3.29]. Since the $\Omega^r(r)$ form a spanning class of \mathbb{P}^{n-1} , we get by adjunction $\text{Hom}^*(E, \mathcal{S}) \neq 0 \neq \text{Hom}^*(\mathcal{S}, E)$. \square

4.2. On the equivariant blow-up. Recall that the blow-up morphism $q: \tilde{X} \rightarrow X$ is G -equivariant. Let $\mathcal{L}_{\tilde{X}} \in \text{Pic}^G(\tilde{X})$ (we will sometimes simply write \mathcal{L} instead of $\mathcal{L}_{\tilde{X}}$) be the equivariant line bundle $\mathcal{O}_{\tilde{X}}(Z)$ equipped with the unique linearisation whose restriction to Z gives the trivial action on $\mathcal{O}_Z(Z) \cong \mathcal{O}_\nu(-1)$. We consider a point $z \in Z$ with $\nu(z) = s$ corresponding to a line $\ell \subset N_{S/X}(s)$. Then the normal space $N_{Z/\tilde{X}}(z)$ can be equivariantly identified with ℓ . It follows by Condition 3.1 that $N_{Z/\tilde{X}} \cong (\mathcal{L}_{\tilde{X}} \otimes \chi^{-1})|_Z$ as an equivariant bundle. Hence, in $\text{Coh}_G(\tilde{X})$, there is the exact sequence

$$(11) \quad 0 \rightarrow \mathcal{L}_{\tilde{X}}^{-1} \otimes \chi \rightarrow \mathcal{O}_{\tilde{X}} \rightarrow \mathcal{O}_Z \rightarrow 0$$

where both $\mathcal{O}_{\tilde{X}}$ and \mathcal{O}_Z are equipped with the canonical linearisation, which is the one given by the trivial action over Z .

Lemma 4.6. *For $\ell = 0, \dots, n-1$ we have $p_* \mathcal{L}_{\tilde{X}}^\ell = \mathcal{O}_X \otimes \chi^\ell$.*

Proof. We have $p_* \mathcal{O}_{\tilde{X}} \cong \mathcal{O}_X$, both, $\mathcal{O}_{\tilde{X}}$ and \mathcal{O}_X , equipped with the canonical linearisations. Hence, the assertion is true for $\ell = 0$. By induction, we may assume that $p_* \mathcal{L}_{\tilde{X}}^{\ell-1} \cong \mathcal{O}_X \otimes \chi^{\ell-1}$. We tensor (11) by $\mathcal{L}_{\tilde{X}}^\ell$ to get

$$0 \rightarrow \mathcal{L}_{\tilde{X}}^{\ell-1} \otimes \chi \rightarrow \mathcal{L}_{\tilde{X}}^\ell \rightarrow \mathcal{O}_\nu(-\ell) \rightarrow 0.$$

Since $0 \leq \ell \leq n-1$, we have $p_* \mathcal{O}_\nu(-\ell) = 0$. Hence, we get an isomorphism

$$p_*(\mathcal{L}_{\tilde{X}}^\ell) \cong p_*(\mathcal{L}_{\tilde{X}}^{\ell-1} \otimes \chi) \cong p_*(\mathcal{L}_{\tilde{X}}^{\ell-1}) \otimes \chi \cong \mathcal{O}_X \otimes \chi^{\ell-1} \otimes \chi \cong \mathcal{O}_X \otimes \chi^\ell. \quad \square$$

Lemma 4.7. *The smooth blow-up $p: \tilde{X} \rightarrow X$ has G -linearised relative dualising sheaf*

$$\omega_p \cong \mathcal{L}_{\tilde{X}}^{n-1} \otimes \chi^{1-n} = \mathcal{L}_{\tilde{X}}^{n-1} \otimes \chi \in \text{Pic}^G(\tilde{X}).$$

Proof. The non-equivariant relative dualising sheaf of the blow-up is $\omega_p \cong \mathcal{O}_{\tilde{X}}((n-1)Z)$. Since p is G -equivariant, there is a unique linearisation of ω_p such that $p^! = p^*(_) \otimes \omega_p: D_G^b(X) \rightarrow D_G^b(\tilde{X})$ is the right-adjoint of $p_*: D_G^b(\tilde{X}) \rightarrow D_G^b(X)$. In the following, we will compute this linearisation of ω_p .

As the equivariant pull-back p^* is fully faithful, $p^!: D_G^b(X) \rightarrow D_G^b(\tilde{X})$ is fully faithful, too. Hence, adjunction gives an isomorphism of equivariant sheaves, $p_* \omega_p \cong p_* p^! \mathcal{O}_X \cong \mathcal{O}_X$. The claim now follows from Lemma 4.6. \square

We denote by $i_s: \mathbb{P}^{n-1} \cong \varrho^{-1}(s) \hookrightarrow \tilde{Y}$ the embedding of the fibre of ϱ and by $j_s: \mathbb{P}^{n-1} \cong p^{-1}(s) \hookrightarrow \tilde{X}$ the embedding of the fibre of p over $s \in S$.

Lemma 4.8. *Let $s \in S$ and $r = 0, \dots, n-1$,*

$$\mathcal{H}^{-r}(p^*\mathcal{O}_s) \cong j_{s*}(\Omega^r(r) \otimes \chi^r).$$

Proof. It is well known that, for the underlying non-equivariant sheaves, we have $\mathcal{H}^{-r}(p^*\mathcal{O}_s) \cong j_{s*}\Omega^r(r)$; see [Huy06, Prop. 11.12]. Since the sheaves $\Omega^r(r)$ are simple, i.e. $\text{End}(\Omega^r(r)) = \mathbb{C}$, we have $\mathcal{H}^{-r}(p^*\mathcal{O}_s) \cong j_{s*}(\Omega^r(r) \otimes \chi^{\alpha_r})$ for some $\alpha_r \in \mathbb{Z}/m\mathbb{Z}$. So we only need to show $\alpha_r = r$.

Let $r \in \{0, \dots, n-1\}$. We have $p_!\mathcal{L}^{-r} \cong p_*(\mathcal{L}^{-r+n-1} \otimes \chi^{1-n})$ by Lemma 4.7. Since $-r+n-1 \in \{0, \dots, n-1\}$, Lemma 4.6 gives $p_!\mathcal{L}^{-r} \cong \mathcal{O}_X \otimes \chi^{-r}$. By adjunction,

$$\mathbb{C}[0] \cong \text{Hom}_{\text{D}_G^b(X)}^*(\mathcal{O}_X \otimes \chi^{-r}, \mathcal{O}_s \otimes \chi^{-r}) \cong \text{Hom}_{\text{D}_G^b(\tilde{X})}^*(\mathcal{L}^{-r}, p^*\mathcal{O}_s \otimes \chi^{-r}).$$

By Lemma 2.3, for $r \neq v$, we have

$$\text{Hom}_{\text{D}_G^b(\tilde{X})}^*(\mathcal{O}_{\tilde{X}}(-rZ), j_{s*}\Omega^v(v)) \cong \text{Hom}_{\text{D}^b(\mathbb{P}^{n-1})}^*(\mathcal{O}(r), \Omega^v(v)) = 0.$$

Using the spectral sequence in $\text{D}_G^b(\tilde{X})$

$$E_2^{u,v} = \text{Hom}^u(\mathcal{L}^{-r}, \mathcal{H}^v(p^*\mathcal{O}_s \otimes \chi^{-r})) \Rightarrow E^{u+v} = \text{Hom}^{u+v}(\mathcal{L}^{-r}, p^*\mathcal{O}_s \otimes \chi^{-r})$$

it follows that

$$\begin{aligned} \mathbb{C}[0] &\cong \text{Hom}_{\text{D}_G^b(\tilde{X})}^*(\mathcal{L}^{-r}, p^*\mathcal{O}_s \otimes \chi^{-r}) \\ &\cong \text{Hom}_{\text{D}_G^b(\tilde{X})}^*(\mathcal{L}^{-r}, \mathcal{H}^{-r}(p^*\mathcal{O}_s) \otimes \chi^{-r})[r] \\ &\cong (\text{Hom}_{\text{D}^b(\mathbb{P}^{n-1})}^*(\mathcal{O}(r), \Omega^r(r)) \otimes \chi^{\alpha_r} \otimes \chi^{-r})^G[r] \\ &\cong (\mathbb{C}[-r] \otimes \chi^{\alpha_r-r})^G[r] \end{aligned}$$

where the last isomorphism is again due to Lemma 2.3. Comparing the first and last term of the above chain of isomorphisms, we get $\mathbb{C} \cong (\chi^{\alpha_r-r})^G$ which implies $\alpha_r = r$. \square

Corollary 4.9. *Let $n \geq m$ and $\ell \in \{0, \dots, m-1\}$. Let $\lambda \geq 0$ be the largest integer such that $\ell + \lambda m \leq n-1$. Then*

$$\mathcal{H}^*(\Psi(\mathcal{O}_s \otimes \chi^{-\ell})) \cong i_{s*}(\bigoplus_{t=0}^{\lambda} \Omega^{\ell+tm}(\ell+tm)[\ell+tm]).$$

Proof. Since the (non-derived) functor $q_*^G: \text{Coh}^G(\tilde{X}) \rightarrow \text{Coh}(\tilde{Y})$ is exact, we have

$$\mathcal{H}^r(\Psi(\mathcal{O}_s \otimes \chi^{-\ell})) \cong q_*^G(\mathcal{H}^{-r}(p^*\mathcal{O}_s) \otimes \chi^{-\ell})$$

and the claim follows from Lemma 4.8. \square

4.3. On the cyclic cover. The morphism $q: \tilde{X} \rightarrow \tilde{Y} = \tilde{X}/G$ is a cyclic cover branched over the divisor Z . This geometric situation and the derived categories involved are studied in great detail in [KP14]. However, we will only need the following basic facts, all of which can be found in [KP14, Sect. 4.1].

Lemma 4.10.

- (i) *The sheaf of invariants $q_*^G(\mathcal{O}_{\tilde{X}} \otimes \chi^{-1})$ is a line bundle which we denote $\mathcal{L}_{\tilde{Y}}^{-1} \in \text{Pic}(\tilde{Y})$.*
- (ii) *$\mathcal{L}_{\tilde{Y}}^m \cong \mathcal{O}_{\tilde{Y}}(Z)$.*
- (iii) *$q_*^G(\mathcal{O}_{\tilde{X}} \otimes \chi^\alpha) \cong \mathcal{L}_{\tilde{Y}}^\alpha$ for $\alpha \in \{-m+1, \dots, 0\}$.*
- (iv) *$q^* \circ \text{triv}: \text{D}^b(\tilde{Y}) \hookrightarrow \text{D}_G^b(\tilde{X})$ is fully faithful, due to $q_*^G(\mathcal{O}_{\tilde{X}}) \cong \mathcal{O}_{\tilde{Y}}$.*

- (v) $q^*(\text{triv}(\mathcal{L}_{\tilde{Y}})) \cong \mathcal{L}_{\tilde{X}}$ are isomorphic G -equivariant line bundles.
 (vi) In particular, $\mathcal{L}_{\tilde{Y}|Z} \cong \mathcal{L}_{\tilde{X}|Z} \cong \mathcal{O}_\nu(-1)$.

Corollary 4.11. $\Psi(\mathcal{O}_X \otimes \chi^\alpha) \cong \mathcal{L}_{\tilde{Y}}^\alpha$ for $\alpha \in \{-m+1, \dots, 0\}$.

Lemma 4.12. The relative dualising sheaf of $q: \tilde{X} \rightarrow \tilde{Y} = \tilde{X}/G$ is $\omega_q \cong \mathcal{O}_{\tilde{X}}((m-1)Z)$.

Proof. Since the G -action on $W := \tilde{X} \setminus Z$ is free, we have $\omega_{q|W} \cong \mathcal{O}_W$. Hence, $\omega_q \cong \mathcal{O}_{\tilde{X}}(\alpha Z)$ for some $\alpha \in \mathbb{Z}$. We have $\mathcal{H}om(\mathcal{O}_Z, \mathcal{O}_{\tilde{X}}) \cong \mathcal{O}_Z(Z)[-1] \cong j_*\mathcal{O}_\nu(-1)[-1]$, and hence

$$\begin{aligned}
 i_*\mathcal{O}_\nu(-1)[-1] &\cong q_*j_*\mathcal{O}_\nu(-1)[-1] \\
 &\cong q_*\mathcal{H}om(\mathcal{O}_Z, \mathcal{O}_{\tilde{X}}) \\
 &\cong q_*\mathcal{H}om(\mathcal{O}_Z, q^*\mathcal{O}_{\tilde{Y}}) \\
 &\cong q_*\mathcal{H}om(\mathcal{O}_Z, q^!\mathcal{L}_{\tilde{Y}}^{-\alpha}) \quad \text{by Lemma 4.10(v)} \\
 &\cong \mathcal{H}om(q_*\mathcal{O}_Z, \mathcal{L}_{\tilde{Y}}^{-\alpha}) \quad \text{by Grothendieck duality} \\
 &\cong \mathcal{O}_Z(Z) \otimes \mathcal{L}_{\tilde{Y}}^{-\alpha}[-1] \\
 &\cong i_*\mathcal{O}_\nu(-m+\alpha)[-1] \quad \text{by Lemma 4.10(ii)+(vi)}
 \end{aligned}$$

and thus we conclude $\alpha = m-1$. \square

Remark 4.13. As an equivariant bundle, we have $\omega_q \cong \mathcal{L}_{\tilde{X}}^{m-1} \otimes \chi$, but we will not use this in the following.

Corollary 4.14. We have $\omega_{\tilde{Y}|Z} \cong \mathcal{O}_\nu(m-n)$.

Proof. We have $\omega_{\tilde{X}|Z} \cong \mathcal{O}_\nu(-n+1)$; compare Lemma 4.7. Furthermore, $\omega_{\tilde{Y}|Z} \cong (q^*\omega_{\tilde{Y}})|_Z$. Hence,

$$\mathcal{O}_\nu(1-m) \stackrel{4.12}{\cong} \omega_{q|Z} \cong \omega_{\tilde{X}|Z} \otimes \omega_{\tilde{Y}|Z}^\vee \cong \mathcal{O}_\nu(1-n) \otimes \omega_{\tilde{Y}|Z}^\vee. \quad \square$$

4.4. The case $m \geq n$. Throughout this subsection, let $m \geq n$.

Proposition 4.15.

- (i) If $m > n$, then the functor Ξ_α is fully faithful for any $\alpha \in \mathbb{Z}/m\mathbb{Z}$.
 (ii) Let $m-n \geq 2$ and $\alpha \neq \beta \in \mathbb{Z}/m\mathbb{Z}$. Then

$$\Xi_\beta^R \Xi_\alpha = 0 \iff \alpha - \beta \in \{\overline{n-m+1}, \overline{n-m+2}, \dots, \overline{-1}\}.$$

Proof. Recall that $\Xi_\beta = (a_* \circ \text{triv}(_)) \otimes \chi^\beta: \text{D}^b(S) \rightarrow \text{D}_G^b(X)$. Hence, the right-adjoint of Ξ_β is given by $\Xi_\beta^R \cong (a^!(_) \otimes \chi^{-\beta})^G$. By [AC12, Thm. 1.4 & Sect. 1.20],

$$\begin{aligned}
 \Xi_\beta^R \Xi_\alpha &\cong (a^!a_*(_) \otimes \chi^{\alpha-\beta})^G \cong ((_) \otimes \wedge^* N \otimes \chi^{\alpha-\beta})^G \\
 &\cong (_) \otimes (\wedge^* N \otimes \chi^{\alpha-\beta})^G
 \end{aligned}$$

where, by Condition 3.1, the G -action on $\wedge^\ell N$ is given by $\chi^{-\ell}$. We see that $(\wedge^* N)^G \cong \wedge^0 N[0] \cong \mathcal{O}_S[0]$; here we use that $m > n$. This shows that, in the case $\alpha = \beta$, we have $\Xi_\alpha^R \Xi_\alpha \cong \text{id}$ which proves (i). Furthermore, since the characters occurring in $\wedge^* N$ are $\chi^0, \chi^{-1}, \dots, \chi^{-n}$, we obtain (ii) from

$$\begin{aligned}
 \Xi_\beta^R \Xi_\alpha \neq 0 &\iff (\wedge^* N \otimes \chi^{\alpha-\beta})^G \neq 0 \iff \overline{0} \in \{\overline{\alpha-\beta}, \overline{\alpha-\beta-1}, \dots, \overline{\alpha-\beta-n}\}, \text{ i.e.} \\
 \Xi_\beta^R \Xi_\alpha = 0 &\iff \alpha - \beta \in \{\overline{n+1}, \dots, \overline{m-1}\} = \{\overline{n-m+1}, \overline{n-m+2}, \dots, \overline{-1}\}. \quad \square
 \end{aligned}$$

Corollary 4.16. *For $m > n$, there is a semi-orthogonal decomposition*

$$\mathrm{D}_G^b(X) = \langle \Xi_{n-m}(\mathrm{D}^b(S)), \Xi_{n-m+1}(\mathrm{D}^b(S)), \dots, \Xi_{-1}(\mathrm{D}^b(S)), \mathcal{A} \rangle,$$

where $\mathcal{A} = {}^\perp \langle \Xi_{n-m}(\mathrm{D}^b(S)), \Xi_{n-m+1}(\mathrm{D}^b(S)), \dots, \Xi_{-1}(\mathrm{D}^b(S)) \rangle$.

Proposition 4.17. *The functor $\Phi = p_* q^* \mathrm{triv}: \mathrm{D}^b(\tilde{Y}) \rightarrow \mathrm{D}_G^b(X)$ is fully faithful.*

Proof. By [Huy06, Prop. 7.1], we only need to show for $x, y \in \tilde{Y}$ that

$$\mathrm{Hom}_{\mathrm{D}_G^b(X)}^i(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_y)) = \begin{cases} \mathbb{C} & \text{if } x = y \text{ and } i = 0 \\ 0 & \text{if } x \neq y \text{ or } i \notin [0, \dim X]. \end{cases}$$

By Proposition 3.2, $\Phi(\mathcal{O}_x) = \mathcal{O}_\xi$ for some G -cluster ξ . Hence,

$$\mathrm{Hom}_{\mathrm{D}_G^b(X)}^0(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_x)) \cong H^0(\mathcal{O}_\xi)^G \cong \mathbb{C}.$$

Furthermore, since $\Phi(\mathcal{O}_x)$ is a sheaf, the complex $\mathrm{Hom}^*(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_x))$ is concentrated in degrees $0, \dots, \dim(X)$. It remains to prove the orthogonality for $x \neq y$. If $\varrho(x) \neq \varrho(y)$, the corresponding G -clusters are supported on different orbits. Hence, their structure sheaves are orthogonal. If $\varrho(x) = \varrho(y)$ but $x \neq y$, the orthogonality was shown in Lemma 3.3. \square

Lemma 4.18. *The functor Φ factors through \mathcal{A} .*

Proof. By Corollary 4.16, this statement is equivalent to $\Phi^R \Xi_\alpha = 0$ for $\alpha \in \{n-m, \dots, -1\}$ where $\Phi^R: \mathrm{D}_G^b(X) \rightarrow \mathrm{D}^b(\tilde{Y})$ is the right adjoint of Φ . Since the composition $\Phi^R \Xi_\alpha$ is a Fourier–Mukai transform, it is sufficient to test the vanishing on skyscraper sheaves of points; see [Kuz06, Sect. 2.2]. So we have to prove that

$$\Phi^R \Xi_\alpha(\mathcal{O}_s) \cong \Phi^R(\mathcal{O}_s \otimes \chi^\alpha) = 0$$

for every $s \in S$ and every $\alpha \in \{n-m, \dots, -1\}$. We have $\Phi^R \cong q_*^G p^!$; recall that q_*^G stands for $(_)^G \circ q_*$. By Lemma 4.7 together with Lemma 4.8, we have

$$\mathcal{H}^{-r}(p^! \mathcal{O}_s) \cong i_{s*}(\Omega^r(r+1-n) \otimes \chi^{r+1-n})$$

where the non-vanishing cohomologies occur for $r \in \{0, \dots, n-1\}$. Thus, the linearisations of the cohomologies of $p^!(\mathcal{O}_s \otimes \chi^\alpha)$ are given by the characters χ^γ for $\gamma \in \{\alpha+1-n, \dots, \alpha\}$. We see that, for $\alpha \in \{n-m, \dots, -1\}$, the trivial character does not occur in $\mathcal{H}^*(p^! \mathcal{O}_s \otimes \chi^\alpha)$. This implies that $q_* p^!(\mathcal{O}_s \otimes \chi^\alpha)$ has vanishing G -invariants. \square

We denote by $\mathcal{B} \subset \mathrm{D}_G^b(X)$ the full subcategory generated by the admissible subcategories $\Xi_\alpha(\mathrm{D}^b(S))$ for $\alpha \in \{n-m, \dots, -1\}$ and $\Phi(\mathrm{D}^b(\tilde{Y}))$. By the above, these admissible subcategories actually form a semi-orthogonal decomposition

$$\mathcal{B} = \langle \Xi_{n-m}(\mathrm{D}^b(S)), \Xi_{n-m+1}(\mathrm{D}^b(S)), \dots, \Xi_{-1}(\mathrm{D}^b(S)) \rangle.$$

Proposition 4.19. *We have the (essential) equalities $\mathcal{B} = \mathrm{D}_G^b(X)$ and $\Phi(\mathrm{D}^b(\tilde{Y})) = \mathcal{A}$.*

For the proof, we need the following

Lemma 4.20. *We have $p_* \mathcal{L}_{\tilde{X}}^r \otimes \chi^{-\lambda} \in \mathcal{B}$ for $r \in \mathbb{Z}$ and $\lambda \in \{0, \dots, m-n\}$.*

Proof. By [Lemma 4.10](#), $\mathcal{L}_{\tilde{X}}^r \cong q^*(\text{triv}(\mathcal{L}_{\tilde{Y}}^r))$. Hence,

$$p_*(\mathcal{L}_{\tilde{X}}^r) \cong p_*q^*(\text{triv}(\mathcal{L}_{\tilde{Y}}^r)) = \Phi(\mathcal{L}_{\tilde{Y}}^r) \in \Phi(D^b(\tilde{Y})) \subset \mathcal{B}$$

which proves the assertion for $\lambda = 0$. We now proceed by induction over λ . Tensoring (11) by $\mathcal{L}_{\tilde{X}}^r \otimes \chi^{-\lambda}$ and applying p_* , we get the exact triangle

$$(12) \quad p_*\mathcal{L}_{\tilde{X}}^{r-1} \otimes \chi^{-(\lambda-1)} \rightarrow p_*\mathcal{L}_{\tilde{X}}^r \otimes \chi^{-\lambda} \rightarrow p_*j_*\mathcal{O}_Z(-r) \otimes \chi^{-\lambda} \rightarrow$$

where $\mathcal{O}_Z(-r)$ carries the trivial G -action.

The first term of the triangle is an object of \mathcal{B} by induction. Furthermore, by diagram (8), we have $p_*j_*\mathcal{O}_Z(-r) \cong a_*\nu_*\mathcal{O}_Z(-r)$. Hence, the third term of (12) is an object of $a_*D_G^b(S) \otimes \chi^{-\lambda} = \Xi_{-\lambda}(D^b(S)) \subset \mathcal{B}$. Thus, also the middle term is an object of \mathcal{B} which gives the assertion. \square

Proof of Proposition 4.19. The second assertion follows from the first one since, if $\mathcal{B} = D_G^b(X)$ holds, both, $\Phi(D^b(\tilde{Y}))$ and \mathcal{A} , are given by the left-orthogonal complement of

$$\langle \Xi_{n-m}(D^b(S)), \Xi_{n-m+1}(D^b(S)), \dots, \Xi_{-1}(D^b(S)) \rangle$$

in $D_G^b(X)$.

The subcategories $\Xi_\alpha(D^b(S))$ and $\Phi(D^b(\tilde{Y}))$ of $D_G^b(X)$ are Y -linear by [Lemma 4.4](#). Hence, for the equality $\mathcal{B} = D_G^b(X)$ it suffices to show that

$$\mathcal{O}_X \otimes \chi^\ell \in \mathcal{B} = \langle D^b(S) \otimes \chi^{n-m}, \dots, D^b(S) \otimes \chi^{-1}, \Phi(D^b(\tilde{Y})) \rangle$$

for every $\ell \in \mathbb{Z}/m\mathbb{Z}$; see [Lemma 4.3](#). Combining [Lemma 4.10](#) and [Lemma 4.6](#), we see that

$$\Phi(\mathcal{L}_{\tilde{Y}}^\ell) \cong p_*(q^*\text{triv}(\mathcal{L}_{\tilde{Y}}^\ell)) \cong \mathcal{O}_X \otimes \chi^\ell \quad \text{for } \ell = 0, \dots, n-1.$$

In particular, $\mathcal{O}_X \otimes \chi^\ell \in \Phi(D^b(\tilde{Y})) \subset \mathcal{B}$ for $\ell = 0, \dots, n-1$. Setting $r = 0$ in the previous lemma, we find that also $\mathcal{O}_X \otimes \chi^\ell$ for $\ell = n-m, \dots, -1$ is an object of \mathcal{B} . \square

Combining the results of this subsection gives [Theorem 4.1\(i\)](#).

4.5. The case $n \geq m$. Throughout this subsection, let $n \geq m$.

Proposition 4.21. *Let $n > m$. Then the functors $\Theta_\beta: D^b(S) \rightarrow D^b(\tilde{Y})$ are fully faithful for every $\beta \in \mathbb{Z}$ and there is a semi-orthogonal decomposition*

$$D^b(\tilde{Y}) = \langle \mathcal{C}(m-n), \mathcal{C}(m-n+1), \dots, \mathcal{C}(-1), \mathcal{D} \rangle$$

where $\mathcal{C}(\ell) := \Theta_\ell(D^b(S)) = i_*\nu^*D^b(S) \otimes \mathcal{O}_{\tilde{Y}}(\ell)$ and

$$\begin{aligned} \mathcal{D} &= \{E \in D^b(\tilde{Y}) \mid i^*E \in {}^\perp \langle \nu^*D^b(S) \otimes \mathcal{O}_{\tilde{Y}}(m-n), \dots, \nu^*D^b(S) \otimes \mathcal{O}_{\tilde{Y}}(-1) \rangle\} \\ &= \{E \in D^b(\tilde{Y}) \mid i^*E \in \langle \nu^*D^b(S), \dots, \nu^*D^b(S) \otimes \mathcal{O}_{\tilde{Y}}(m-1) \rangle\}. \end{aligned}$$

Proof. This follows from [[Kuz08](#), Thm. 1]. However, for convenience, we provide a proof for our special case. By construction, $\Theta_\beta^R \cong \nu_*M_{\mathcal{O}_\nu(-\beta)}i^!$. We start with the standard exact triangle of functors $\text{id} \rightarrow i^!i_* \rightarrow M_{\mathcal{O}_Z(Z)}[-1] \rightarrow$ (see e.g. [[Huy06](#), Cor. 11.4]). By [Lemma 4.10](#), $\mathcal{O}_Z(Z) \cong \mathcal{O}_\nu(-m)$, and thus the above triangle induces for any $\alpha, \beta \in \mathbb{Z}$

$$\nu_*M_{\mathcal{O}_\nu(\alpha-\beta)}\nu^* \rightarrow \Theta_\beta^R\Theta_\alpha \rightarrow \nu_*M_{\mathcal{O}_\nu(\alpha-\beta-m)}\nu^* \rightarrow .$$

By projection formula, we can rewrite this as

$$(_) \otimes \nu_* \mathcal{O}_\nu(\alpha - \beta) \rightarrow \Theta_\beta^R \Theta_\alpha \rightarrow (_) \otimes \nu_* \mathcal{O}_\nu(\alpha - \beta - m) \rightarrow .$$

Now, $\nu_* \mathcal{O}_\nu \cong \mathcal{O}_S$ and $\nu_* \mathcal{O}_\nu(\gamma) = 0$ for $\gamma \in \{-n+1, \dots, -1\}$. Hence, $\Theta_\beta^R \Theta_\beta \cong \text{id}$ and $\Theta_\beta^R \Theta_\alpha = 0$ if $\alpha - \beta \in \{m-n+1, \dots, -1\}$. Therefore, we get a semi-orthogonal decomposition

$$\text{D}^b(\tilde{Y}) = \langle \mathcal{C}(m-n), \mathcal{C}(m-n+1), \dots, \mathcal{C}(-1), \mathcal{D} \rangle .$$

The description of the left-orthogonal \mathcal{D} follows by the adjunction $i^* \dashv i_*$. \square

Lemma 4.22. *The functor $\Psi: \text{D}_G^b(X) \rightarrow \text{D}^b(\tilde{Y})$ factors through \mathcal{D} .*

Proof. By Lemma 4.3, the equivariant bundles $\mathcal{O}_X, \mathcal{O}_X \otimes \chi, \dots, \mathcal{O}_X \otimes \chi^{m-1}$ generate $\text{D}_G^b(X)$ over $\text{D}^{\text{perf}}(Y)$, and therefore so do the bundles $\mathcal{O}_X \otimes \chi^{-m+1}, \dots, \mathcal{O}_X \otimes \chi^{-1}, \mathcal{O}_X$ obtained by twisting with χ^{1-m} . Hence, it is sufficient to prove that $\Psi(\mathcal{O}_X \otimes \chi^\alpha) \in \mathcal{D}$ for $\alpha \in \{-m+1, \dots, 0\}$ as Ψ and \mathcal{D} are Y -linear; see Lemma 4.4.

Indeed, by Lemma 4.10 we have $i^* \mathcal{L}_Y^\alpha = \mathcal{L}_{\tilde{Y}|Z}^\alpha \cong \mathcal{O}_\nu(-\alpha)$, hence

$$\Psi(\mathcal{O}_X \otimes \chi^\alpha) \stackrel{4.11}{\cong} q_*^G(\mathcal{O}_X \otimes \chi^\alpha) \cong \mathcal{L}_{\tilde{Y}}^\alpha \in \mathcal{D} \quad \text{for } \alpha \in \{-m+1, \dots, 0\}. \quad \square$$

Proposition 4.23. *The functor $\Psi: \text{D}_G^b(X) \rightarrow \text{D}^b(\tilde{Y})$ is fully faithful.*

Proof. We first observe that $V := \mathcal{O}_X \otimes \mathbb{C}[G] = \mathcal{O}_X \otimes (\chi^0 \oplus \dots \oplus \chi^{m-1})$ is a relative tilting bundle for $\text{D}_G^b(X)$ over $\text{D}^{\text{perf}}(Y)$; see Lemma 4.3.

For the fully faithfulness, we follow Lemma 2.10. So we need to show that Ψ induces an isomorphism $\Lambda_V = \pi_*^G \mathcal{H}om(V, V) \xrightarrow{\sim} \varrho_* \mathcal{H}om(\Psi(V), \Psi(V))$. In turn, it suffices to consider the direct summands of V . Thus, let $\alpha, \beta \in \{-m+1, \dots, 0\}$ and compute

$$\begin{aligned} \pi_*^G \mathcal{H}om^*(\mathcal{O}_X \otimes \chi^\alpha, \mathcal{O}_X \otimes \chi^\beta) &\cong \pi_*^G \mathcal{H}om^*(\mathcal{O}_X \otimes \chi^{\alpha+n-1} \otimes \chi^{1-n}, \mathcal{O}_X \otimes \chi^\beta) \\ &\stackrel{4.6}{\cong} \pi_*^G \mathcal{H}om^*(p_* \mathcal{L}_{\tilde{X}}^{\alpha+n-1} \otimes \chi^{1-n}, \mathcal{O}_X \otimes \chi^\beta) \\ &\stackrel{4.7}{\cong} \pi_*^G \mathcal{H}om^*(p_*(\mathcal{L}_{\tilde{X}}^\alpha \otimes \omega_p), \mathcal{O}_X \otimes \chi^\beta) \\ &\cong \pi_*^G p_* \mathcal{H}om_X^*(\mathcal{L}_{\tilde{X}}^\alpha \otimes \omega_p, p^! \mathcal{O}_X \otimes \chi^\beta) \\ &\cong \pi_*^G p_* \mathcal{H}om_{\tilde{X}}^*(\mathcal{L}_{\tilde{X}}^\alpha, p^* \mathcal{O}_X \otimes \chi^\beta) \\ &\stackrel{4.10}{\cong} \varrho_* q_*^G \mathcal{H}om_{\tilde{X}}^*(q^* q_*^G(\mathcal{O}_{\tilde{X}} \otimes \chi^\alpha), \mathcal{O}_{\tilde{X}} \otimes \chi^\beta) \\ &\cong \varrho_* \mathcal{H}om_{\tilde{Y}}^*(q_*^G(\mathcal{O}_{\tilde{X}} \otimes \chi^\alpha), q_*^G(\mathcal{O}_{\tilde{X}} \otimes \chi^\beta)) \\ &= \varrho_* \mathcal{H}om_{\tilde{Y}}^*(\Psi(\mathcal{O}_{\tilde{X}} \otimes \chi^\alpha), \Psi(\mathcal{O}_{\tilde{X}} \otimes \chi^\beta)). \quad \square \end{aligned}$$

We denote by $\mathcal{E} \subset \text{D}^b(\tilde{Y})$ the full subcategory generated by the admissible subcategories $\Psi(\text{D}_G^b(X))$ and $\Theta_\ell(\text{D}^b(S)) = i_* \nu^* \text{D}^b(S) \otimes \mathcal{O}_{\tilde{Y}}(\ell)$ for $\ell \in \{m-n, \dots, -1\}$. By the above, these admissible subcategories actually form a semi-orthogonal decomposition

$$\mathcal{E} = \langle \Theta_{m-n}(\text{D}^b(S)), \dots, \Theta_{-1}(\text{D}^b(S)), \Psi(\text{D}_G^b(X)) \rangle \subseteq \text{D}^b(\tilde{Y}).$$

Proposition 4.24. *We have the (essential) equalities $\mathcal{E} = \text{D}^b(\tilde{Y})$ and $\Psi(\text{D}_G^b(X)) = \mathcal{D}$.*

Proof. Analogously to [Proposition 4.19](#), it is sufficient to prove the equality $\mathcal{E} = \mathrm{D}^b(\tilde{Y})$. As \mathcal{E} is constructed from images of fully faithful FM transforms (which have both adjoints), it is admissible in $\mathrm{D}^b(\tilde{Y})$. Therefore, it suffices to show that \mathcal{E} contains a spanning class for $\mathrm{D}^b(\tilde{Y})$. Moreover, because all functors and categories involved are Y -linear, it suffices to prove that the relative spanning class \mathcal{S} of [Lemma 4.5](#) is contained in \mathcal{E} .

We already know that $\mathcal{O}_{\tilde{Y}} \cong \Psi(\mathcal{O}_X) \in \Psi(\mathrm{D}_G^b(X)) \subset \mathcal{E}$. By [Corollary 2.4](#), we get for $s \in S$ and $r \in \{m, \dots, n-1\}$

$$i_{s*}\Omega^r(r) \in \langle \Theta_{m-n}(\mathrm{D}^b(S)), \dots, \Theta_{-1}(\mathrm{D}^b(S)) \rangle \subset \mathcal{E}.$$

By [Corollary 4.9](#), we have, for $\ell \in \{0, \dots, m-1\}$, an exact triangle

$$E \rightarrow \Psi(\mathcal{O}_s \otimes \chi^{-\ell}) \rightarrow i_{s*}\Omega^\ell(\ell)[\ell] \rightarrow$$

where E is an object in the triangulated category spanned by $i_{s*}\Omega^r(r)$ for $r \in \{m, \dots, n-1\}$. In particular, the first two terms of the exact triangle are objects in \mathcal{E} . Hence also $i_{s*}\Omega^\ell(\ell) \in \mathcal{E}$ for $\ell \in \{0, \dots, m-1\}$. \square

Combining the results of this subsection gives [Theorem 4.1\(ii\)](#).

4.6. The case $m = n$: spherical twists and induced tensor products. Throughout this section, let $m = n$. In this case, both functors Φ and Ψ are equivalences. In this subsection, we will show that the functors Θ_β and Ξ_α , which were fully faithful in the cases $n > m$ and $m > n$, respectively, are now spherical. Furthermore, the spherical twists along these functors allow to describe the transfer of the tensor structure from one side of the derived McKay correspondence to the other. We set $\Theta := \Theta_0$ and $\Xi := \Xi_0$.

Proposition 4.25. *For every $\alpha \in \mathbb{Z}/m\mathbb{Z}$, the functor $\Xi_\alpha: \mathrm{D}^b(S) \rightarrow \mathrm{D}_G^b(X)$ is a split spherical functor with cotwist $\mathrm{M}_{\omega_{S/X}}[-n]$.*

Proof. Since $\Xi_\alpha \cong \mathrm{M}_{\chi^\alpha} \Xi$, it is sufficient to prove the assertion for $\alpha = 0$; see [Lemma 2.14](#). Following the proof of [Proposition 4.15](#), we have $\Xi^R \Xi \cong (_) \otimes (\wedge^* N)^G$ where G acts on $\wedge^\ell N$ by $\chi^{-\ell}$. From $\mathrm{rank} N = n = m = \mathrm{ord} \chi$, we get

$$(\wedge^* N)^G \cong \mathcal{O}_S[0] \oplus \det N[-n] \cong \mathcal{O}_S[0] \oplus \omega_{S/X}[-n].$$

Hence, $\Xi^R \Xi \cong \mathrm{id} \oplus C$ with $C := \mathrm{M}_{\omega_{S/X}}[-n]$. Moreover, we have $a^! \cong Ca^*$ which implies $\Xi^R \Xi \cong C \Xi^L$. \square

We introduce autoequivalences $\mathrm{M}_\mathcal{L}: \mathrm{D}^b(\tilde{Y}) \rightarrow \mathrm{D}^b(\tilde{Y})$ and $\mathrm{M}_\chi: \mathrm{D}_G^b(X) \rightarrow \mathrm{D}_G^b(X)$ given by the tensor products with the line bundle $\mathcal{L}_{\tilde{Y}}$ and the character χ , respectively.

Theorem 4.26. *There are the following relations between functors:*

- (i) $\Psi^{-1} \cong \mathrm{M}_\chi \Phi \mathrm{M}_{\mathcal{L}^{n-1}}$;
- (ii) $\Psi \Xi \cong \Theta$, in particular, the functors Θ_β are spherical too;
- (iii) $\mathrm{T}_\Theta \cong \Psi \mathrm{T}_\Xi \Psi^{-1}$;
- (iv) $\Psi^{-1} \mathrm{M}_\mathcal{L} \Psi \cong \mathrm{M}_\chi \mathrm{T}_\Xi$ and $\Psi^{-1} \mathrm{M}_{\mathcal{L}^{-1}} \Psi \cong \mathrm{T}_\Xi^{-1} \mathrm{M}_{\chi^{-1}}$.

Proof. To verify (i), note that

$$\Psi^{-1} \cong \Psi^L \cong p_! q^* \stackrel{4.7}{\cong} p_* \mathrm{M}_{\mathcal{L}_{\tilde{X}}^{n-1} \otimes \chi} q^* \stackrel{4.10}{\cong} \mathrm{M}_\chi p_* q^* \mathrm{M}_{\mathcal{L}^{n-1}} \cong \mathrm{M}_\chi \Phi \mathrm{M}_{\mathcal{L}^{n-1}}.$$

For (ii), first note that, since the G -action on $Z \subset \tilde{X}$ is trivial, we have

$$\Theta \cong i_* \nu^* \cong q_* j_* \nu^* \cong q_*^G j_* \nu^* \text{triv}.$$

Hence, the base change morphism $\vartheta: p^* a_* \rightarrow j_* \nu^*$ induces a morphism of functors

$$\hat{\vartheta}: \Psi \Xi \cong q_*^G p^* a_* \text{triv} \rightarrow q_*^G j_* \nu^* \text{triv} \cong \Theta$$

which in turn is induced by a morphism between the Fourier–Mukai kernels; see [Kuz06, Sect. 2.4]. Hence, it is sufficient to show that ϑ induces an isomorphism $\Psi \Xi(\mathcal{O}_s) \cong \Theta(\mathcal{O}_s)$ for every $s \in S$; see [Kuz06, Sect. 2.2]. The morphism ϑ induces an isomorphism on degree zero cohomology $L^0 p^* a_*(\mathcal{O}_s) \cong \mathcal{O}_{p^{-1}(a(s))} \cong j_* L^0 \nu^*(\mathcal{O}_s)$. But there are no cohomologies in non-zero degrees for $j_* \nu^*$ since ν is flat and j a closed embedding. Furthermore, the non-zero cohomologies of $p^* a_*$ vanish after taking invariants; see Corollary 4.9. Hence, $\hat{\vartheta}(\mathcal{O}_s)$ is indeed an isomorphism.

The second assertion of (ii) and (iii) are direct consequences of Proposition 4.25 and the formula $\Psi \Xi \cong \Theta$; see Lemma 2.14.

For (iv), it is sufficient to prove the second relation, and we employ Corollary 2.12 with $L_i = \mathcal{O}_X \otimes \chi^i$; see also Lemma 4.3. Recall that $\mathbb{T}_{\Xi}^{-1} = \text{cone}(\text{id} \rightarrow \Xi \Xi^L)[-1]$, and $\Xi^L \cong (_)^G a^*$. For $1 \neq \alpha \in \mathbb{Z}/n\mathbb{Z}$, we get

$$\Xi^L M_{\chi^{-1}}(\mathcal{O}_X \otimes \chi^\alpha) \cong (\mathcal{O}_S \otimes \chi^{\alpha-1})^G \cong 0.$$

Hence, $\mathbb{T}_{\Xi} M_{\chi^{-1}}(\mathcal{O}_X \otimes \chi^\alpha) \cong \mathcal{O}_X \otimes \chi^{\alpha-1}$. We have $\Xi^L(\mathcal{O}_X) = \mathcal{O}_S$. Therefore, $\Xi \Xi^L(\mathcal{O}_X) \cong a_* \mathcal{O}_S$ and $\mathbb{T}_{\Xi}(\mathcal{O}_X) \cong \mathcal{I}_S$. In summary,

$$\mathbb{T}_{\Xi}^{-1} M_{\chi^{-1}}(\mathcal{O}_X \otimes \chi^\alpha) \cong \begin{cases} \mathcal{O}_X \otimes \chi^\alpha & \text{for } \alpha \neq 1, \\ \mathcal{I}_S & \text{for } \alpha = 1. \end{cases}$$

On the other hand, for $\alpha \in \{-n+1, \dots, 0\}$, we have $\Psi(\mathcal{O}_X \otimes \chi^\alpha) \cong \mathcal{L}^\alpha$; see Corollary 4.11. Hence, we have

$$\Psi^{-1} M_{\mathcal{L}^{-1}} \Psi(\mathcal{O}_X \otimes \chi^\alpha) \cong \mathcal{O}_X \otimes \chi^{\alpha-1} \quad \text{for } \alpha \in \{-n+2, \dots, 0\}.$$

For $\alpha = -n+1$, we use (i) to get

$$\Psi^{-1} M_{\mathcal{L}^{-1}} \Psi(\mathcal{O}_X \otimes \chi^{1-n}) \cong \Psi^{-1}(\mathcal{L}_{\tilde{Y}}^{-n}) \cong M_\chi \Phi(\mathcal{L}_{\tilde{Y}}^{-1}) \stackrel{4.10}{\cong} p_*(\mathcal{L}_{\tilde{X}}^{-1} \otimes \chi) \cong \mathcal{I}_S$$

where we get the last isomorphism by applying p_* to the exact sequence (11).

Therefore, we obtain isomorphisms

$$\kappa_\alpha: F_1(L_\alpha) := \mathbb{T}_{\Xi}^{-1} M_{\chi^{-1}}(\mathcal{O}_X \otimes \chi^\alpha) \xrightarrow{\sim} F_2(L_\alpha) := \Psi^{-1} M_{\mathcal{L}^{-1}} \Psi(\mathcal{O}_X \otimes \chi^\alpha)$$

for every $\alpha \in \mathbb{Z}/n\mathbb{Z}$.

Finally, we have to check that the isomorphisms κ_α can be chosen in such a way that they form an isomorphism of functors $\kappa: F_{1,V|\{L_0, \dots, L_{n-1}\}} \xrightarrow{\sim} F_{2,V|\{L_0, \dots, L_{n-1}\}}$ over every open set $V \subset Y$. Let $U := Y \setminus S \subset Y$ the open complement of the singular locus. We claim that $F_{1,U} \cong M_\chi^{-1} \cong F_{2,U}$. This is clear for $F_2 = \mathbb{T}_{\Xi}^{-1} M_\chi^{-1}$. Furthermore, the map $p: \tilde{X} \rightarrow X$ is an isomorphism and $q: \tilde{X} \rightarrow \tilde{Y}$ is a free quotient when restricted to $W := \pi^{-1}(U)$. Since also $\mathcal{L}_{\tilde{X}} = q_*^G(\mathcal{O}_{\tilde{X}} \otimes \chi)$, we get $\Psi_U \cong M_\chi^{-1}|_U$.

Hence, over W , the $\kappa_i|_W$ can be chosen functorially. By the above computations, each $\kappa_i|_W$ is given by a section of the trivial line bundle. As S has codimension at least 2 in X , the sections $\kappa_i|_W$ over W uniquely extend to sections κ_i over X . The commutativity of the

diagrams relevant for the functoriality now follows from the commutativity of the diagrams restricted to the dense subset W . \square

The relations of [Theorem 4.26](#) allow to transfer structures between $D^b(\tilde{Y})$ and $D_G^b(X)$. For example, we can deduce the formula $\Psi M_{\chi^{-1}} \Psi^{-1} \cong T_{\Theta} M_{\mathcal{L}^{-1}}$. Since $\mathcal{O}_X \otimes \chi^\alpha$ for $\alpha \in \{-(n-1), \dots, 0\}$ form a relative generator of $D_G^b(X)$, their images \mathcal{L}^α under Ψ do as well. Hence, at least theoretically, our formulas give a complete description of the tensor products induced by Ψ (and also Φ) on both sides.

Note that Φ and Ψ are both equivalences, but not inverse to each other. Hence, they induce non-trivial autoequivalences $\Psi\Phi \in \text{Aut}(D_G^b(X))$ and $\Phi\Psi \in \text{Aut}(D^b(\tilde{Y}))$. Considering the setup of the McKay correspondence as a flop of orbifolds as in diagram (1), it makes sense to call them *flop-flop autoequivalences*. These kinds of autoequivalences were widely studied for flops of varieties; see [Tod07], [DW16], [DW15], [ADM15]. The general picture seems to be that the flop-flop autoequivalences can be expressed via spherical and \mathbb{P} -twists induced by functors naturally associated to the centres of the flops. This picture is called the 'flop-flop=twist' principle; see [ADM15]. The following can be seen as the first instance of an orbifold 'flop-flop=twist' principle which we expect to hold in greater generality.

Corollary 4.27. $\Psi\Phi \cong T_{\Theta} M_{\mathcal{L}^{-n}} \cong T_{\Theta} M_{\mathcal{O}_{\tilde{Y}}(-Z)}$.

Remark 4.28. Let us assume $m = n = 2$ so that $\chi^{-1} = \chi$. Then, for every $k \in \mathbb{N}$, we get

$$(13) \quad \Phi(\mathcal{L}^{-k}) \cong \mathcal{I}_S^k \otimes \chi^k$$

where \mathcal{I}_S^k denotes the k -th power of the ideal sheaf of the fixed point locus. Indeed,

$$\begin{aligned} \Phi(\mathcal{L}^{-k}) &\stackrel{4.26(i)}{\cong} M_{\chi} \Psi^{-1}(\mathcal{L}^{-k-1}) && \cong && M_{\chi}(\Psi^{-1} M_{\mathcal{L}^{-1}} \Psi)^k(\mathcal{L}^{-1}) \\ &&& \stackrel{4.11}{\cong} && M_{\chi}(\Psi^{-1} M_{\mathcal{L}^{-1}} \Psi)^k(\mathcal{O} \otimes \chi) \\ &&& \stackrel{4.26(iv)}{\cong} && (M_{\chi} T_{\Xi}^{-1})^k(\mathcal{O}_X) \\ &&& \cong && \mathcal{I}_S^k \otimes \chi^k. \end{aligned}$$

The last isomorphism follows inductively using the short exact sequences

$$0 \rightarrow \mathcal{I}_S^{k+1} \rightarrow \mathcal{I}_S^k \rightarrow \mathcal{I}_S^k / \mathcal{I}_S^{k+1} \rightarrow 0$$

and the fact that the natural action of μ_2 on $\mathcal{I}_S^k / \mathcal{I}_S^{k+1}$ is given by χ^k . Let now S be a surface and $X = S^2$ with μ_2 acting by permutation of the factors. Then $\tilde{Y} = S^{[2]}$ is the Hilbert scheme of two points and $\mathcal{L}_{\tilde{Y}}$ is the square root of the boundary divisor Z parametrising double points. For a vector bundle F on S of rank r , we have

$$\det F^{[2]} \cong \mathcal{L}_{\tilde{Y}}^{-r} \otimes \mathcal{D}_{\det F}$$

where $F^{[2]}$ denotes the tautological rank $2r$ bundle induced by F and, for $L \in \text{Pic } S$, we put $\mathcal{D}_L := \varrho^* \pi_*(L \boxtimes L)^G \in \text{Pic } S^{[2]}$. Hence, by the \mathcal{O}_Y -linearity of Φ , formula (13) recovers the $n = 2$ case of [Sca15, Thm. 1.8].

5. CATEGORICAL RESOLUTIONS

5.1. General definitions. Recall from [Kuz08] that a *categorical resolution* of a triangulated category \mathcal{T} is a smooth triangulated category $\tilde{\mathcal{T}}$ together with a pair of functors $P_*: \tilde{\mathcal{T}} \rightarrow \mathcal{T}$ and $P^*: \mathcal{T}^{\text{perf}} \rightarrow \tilde{\mathcal{T}}$ such that P^* is left adjoint to P_* on $\mathcal{T}^{\text{perf}}$ and the natural morphism of functors $\text{id}_{\mathcal{T}^{\text{perf}}} \rightarrow P_*P^*$ is an isomorphism. Here, $\mathcal{T}^{\text{perf}}$ is the triangulated category of perfect objects in \mathcal{T} . Moreover, a categorical resolution $(\tilde{\mathcal{T}}, P_*, P^*)$ is *weakly crepant* if the functor P^* is also right adjoint to P_* on $\mathcal{T}^{\text{perf}}$.

For the notion of smoothness of a triangulated category see e.g. [KL15]. For us it is sufficient to notice that every admissible subcategory of $D^b(Z)$ for some smooth variety Z is smooth. In fact, we will always consider categorical resolutions of $D^b(Y)$, for some variety Y with rational Gorenstein singularities, *inside* $D^b(\tilde{Y})$ for some fixed (geometric) resolution of singularities $\varrho: \tilde{Y} \rightarrow Y$. By this we mean an admissible subcategory $\tilde{\mathcal{T}} \subset D^b(\tilde{Y})$ such that $\varrho^*: D^{\text{perf}}(Y) \rightarrow D^b(\tilde{Y})$ factorises through $\tilde{\mathcal{T}}$.

By Grothendieck duality, we get a canonical isomorphism $\mathcal{O}_Y \cong \varrho_*\mathcal{O}_{\tilde{Y}} \cong \varrho_*\omega_{\varrho}$. This induces a global section s of ω_{ϱ} , unique up to scalar multiplication by $\mathcal{O}_Y(Y)^\times$, and hence a morphism of functors

$$t := \varrho_*(- \otimes s): \varrho_* \rightarrow \varrho_!.$$

Since this morphism can be found between the corresponding Fourier–Mukai kernels, we may define the cone of functors $\varrho_+ := \text{cone}(t): D^b(\tilde{Y}) \rightarrow D^b(Y)$.

Definition 5.1. The *weakly crepant neighbourhood of Y inside $D^b(\tilde{Y})$* is the full triangulated subcategory

$$\text{WCN}(\varrho) := \ker(\varrho_+) \subset D^b(\tilde{Y}).$$

Proposition 5.2. *If $\text{WCN}(\varrho)$ is a smooth category (which is the case if it is an admissible subcategory of $D^b(\tilde{Y})$), it is a categorical weakly crepant resolution of singularities.*

Proof. By adjunction formula, $t\varrho^*: \varrho_*\varrho^* \rightarrow \varrho_!\varrho^*$ is an isomorphism. Hence, $\varrho_+\varrho^* = 0$ and $\varrho^*: D^{\text{perf}}(Y) \rightarrow D^b(\tilde{Y})$ factors through $\text{WCN}(\varrho)$. By definition, $\varrho_!$ is the left adjoint to ϱ^* . Since ϱ_* and $\varrho_!$ agree on $\text{WCN}(\varrho)$, we also have the adjunction $\varrho_* \dashv \varrho^*$ on $\text{WCN}(\varrho)$. \square

Remark 5.3. We think of $\text{WCN}(\varrho)$ as the biggest weakly crepant categorical resolution inside the derived category $D^b(\tilde{Y})$ of a given geometric resolution $\varrho: \tilde{Y} \rightarrow Y$. The only thing that prevents us from turning this intuition into a statement is the possibility that, for a given weakly crepant resolution $\mathcal{T} \subset D^b(\tilde{Y})$, there might be an isomorphism $\varrho_{*|\mathcal{T}} \cong \varrho_{!|\mathcal{T}}$ which is not the restriction of t (up to scalars).

5.2. The weakly crepant neighbourhood in the cyclic setup. In the case of the resolution of the cyclic quotient singularities discussed in the earlier sections, $\text{WCN}(\varrho)$ is indeed a categorical resolution by the following result. We use the notation of Section 3; recall $G = \mu_m$.

Theorem 5.4. *Let $Y = X/G$, $\varrho: \tilde{Y} \rightarrow Y$ and $i: Z = \varrho^{-1}(S) \hookrightarrow \tilde{Y}$ be as in Section 3. Assume $m \mid n = \text{codim}(S \hookrightarrow X)$ and $n > m$. Then there is a semi-orthogonal decomposition*

$$\text{WCN}(\varrho) = \langle i_*(\mathcal{E}), \Psi(D_{\mu_m}^b(X)) \rangle$$

where

$$\begin{aligned} \mathcal{E} = & \langle \mathcal{A}(-m+1), \mathcal{A}(-m+2), \dots, \mathcal{A}(-1), \\ & \mathcal{A} \otimes \Omega^{n-m-1}(n-m-1), \mathcal{A} \otimes \Omega^{n-m-2}(n-m-2), \dots, \mathcal{A} \otimes \Omega^m(m) \rangle. \end{aligned}$$

with $\mathcal{A} := \nu^* \mathrm{D}^b(S)$ and $\mathcal{A}(i) := \mathcal{A} \otimes \mathcal{O}(i)$; the $\mathcal{A} \otimes \Omega^i(i)$ parts of the decomposition do not occur for $n = 2m$. In particular, $\mathrm{WCN}(\varrho)$ is an admissible subcategory of $\mathrm{D}^b(\tilde{Y})$.

Proof. We first want to show that $\Psi(\mathrm{D}_{\mu_m}^b(X)) \subset \mathrm{WCN}(\varrho)$. For this, by Lemma 4.3, it is sufficient to show that $\mathcal{L}_{\tilde{Y}}^a = \Psi(\mathcal{O}_X \otimes \chi^a) \in \mathrm{WCN}(\varrho)$ for every $a \in \{-m+1, \dots, 0\}$.

The equivariant derived category $\mathrm{D}_{\mu_m}^b(X)$ is a strongly (hence also weakly) crepant categorical resolution of the singularities of Y via the functors $\pi^*, \pi_*^{\mu_m}$; see [Abu16, Thm. 1.0.2]. Since $\Psi \circ \pi^* \cong \varrho^*$ (see Lemma 4.4), $\mathcal{C} := \Psi(\mathrm{D}_{\mu_m}^b(X))$ is a crepant resolution via the functors ϱ^*, ϱ_* . Hence, $\varrho_* \mathcal{L}_{\tilde{Y}}^a \cong \varrho_! \mathcal{L}_{\tilde{Y}}^a$ for $a \in \{-m+1, \dots, 0\}$ and it is only left to show that this isomorphism is induced by t . Again by the Y -linearity of Ψ , we have $\varrho_* \mathcal{L}_{\tilde{Y}}^a \cong \pi_*(\mathcal{O}_X \otimes \chi^a)^{\mu_m}$ which is a reflexive sheaf on the normal variety Y (this follows for example by [Har80, Cor. 1.7]). By construction, t induces an isomorphism over $Y \setminus S$. Since the codimension of S is at least 2, $t: \varrho_* \mathcal{L}_{\tilde{Y}}^a \rightarrow \varrho_! \mathcal{L}_{\tilde{Y}}^a$ is an isomorphism of reflexive sheaves over all of Y ; see [Har80, Prop. 1.6].

By Theorem 4.1(ii), we have $\mathrm{D}^b(\tilde{Y}) \cong \langle \mathcal{B}, \mathcal{C} \rangle$ with

$$\mathcal{B} \cong i_* \langle \mathcal{A}(m-n), \dots, \mathcal{A}(-1) \rangle \cong i_* (\langle \mathcal{A}, \mathcal{A}(1), \dots, \mathcal{A}(m-1) \rangle^\perp).$$

We have $\varrho_* \mathcal{B} = 0$. It follows that $\mathrm{WCN}(\varrho) = \langle \mathcal{B} \cap \ker(\varrho_!), \mathcal{C} \rangle$. Indeed, consider an object $A \in \mathrm{D}^b(\tilde{Y})$. It fits into an exact triangle $C \rightarrow A \rightarrow B \rightarrow$ with $C \in \mathcal{C}$ and $B \in \mathcal{B}$. From the morphism of triangles

$$\begin{array}{ccccccc} \varrho_*(C) & \longrightarrow & \varrho_*(A) & \longrightarrow & \varrho_*(B) = 0 & \longrightarrow & \\ \cong \downarrow t(C) & & \downarrow t(A) & & \downarrow t(B) & & \\ \varrho_!(C) & \longrightarrow & \varrho_!(A) & \longrightarrow & \varrho_!(B) & \longrightarrow & \end{array}$$

we see that $t(A)$ is an isomorphism if and only if $\varrho_! B = 0$.

It is left to compute $\mathcal{B} \cap \ker(\varrho_!)$. Let $F \in \mathrm{D}^b(Z)$ and $B = i_* F$. By Corollary 4.14,

$$\varrho_! B \cong \varrho_! i_* F \cong b_* \nu_*(F \otimes \mathcal{O}_\nu(m-n)).$$

We see that $B \in \ker \varrho_!$ if and only if $\nu_*(F \otimes \mathcal{O}_\nu(m-n)) = 0$ if and only if $F \in \nu^* \mathrm{D}^b(S)(n-m)^\perp$. Hence, $\mathcal{B} \cap \ker \varrho_! = i_*(\mathcal{F}^\perp)$ with

$$\mathcal{F} = \langle \mathcal{A}, \mathcal{A}(1), \dots, \mathcal{A}(m-1), \mathcal{A}(n-m) \rangle \subset \mathrm{D}^b(Z)$$

Carrying out the appropriate mutations within the semi-orthogonal decomposition

$$\mathrm{D}^b(Z) = \langle \mathcal{A}(-m+1), \mathcal{A}(-m+2), \dots, \mathcal{A}(n-m-1), \mathcal{A}(n-m) \rangle,$$

we see that $\mathcal{F}^\perp = \mathcal{E}$; compare Lemma 2.3.

Since $\mathcal{E} \subset \langle \mathcal{A}(m-n), \dots, \mathcal{A}(-1) \rangle$ is an admissible subcategory, we find that $i_*: \mathcal{E} \rightarrow \mathrm{D}^b(\tilde{Y})$ is fully faithful and has adjoints. Hence, $\mathrm{WCN}(\varrho) \subset \mathrm{D}^b(\tilde{Y})$ is admissible. \square

Remark 5.5. We have $\mathrm{D}^b(\tilde{Y}) = \langle \mathcal{A} \otimes \Omega^{n-m}(n-m), \mathrm{WCN}(\varrho) \rangle$. In other words, we can achieve categorical weak crepancy by dropping only one $\mathrm{D}^b(S)$ part of the semi-orthogonal decomposition of $\mathrm{D}^b(\tilde{Y})$.

5.3. The discrepant category and some speculation. Let Y be a variety with rational Gorenstein singularities and $\varrho: \tilde{Y} \rightarrow Y$ a resolution of singularities. Then, ϱ is a crepant resolution if and only if $D^b(\tilde{Y}) = \text{WCN}(\varrho)$; compare [Abu16, Prop. 2.0.10]. We define the *discrepant category* of the resolution as the Verdier quotient

$$\text{disc}(\varrho) := D^b(\tilde{Y}) / \text{WCN}(\varrho).$$

By [Nee01, Remark 2.1.10], since $\text{WCN}(\varrho)$ is a kernel, and hence a thick subcategory, we have $\text{disc}(\varrho) = 0$ if and only if $D^b(\tilde{Y}) = \text{WCN}(\varrho)$. Therefore, we can regard $\text{disc}(\varrho)$ as a categorical measure of the discrepancy of the resolution $\varrho: \tilde{Y} \rightarrow Y$.

In our cyclic quotient setup, where $\tilde{Y} \cong \text{Hilb}^G(X)$ is the simple blow-up resolution, we have $\text{disc}(\varrho) \cong D^b(S)$ by Remark 5.5 and [LS16, Lem. A.8]. Hence, in this case, $\text{disc}(\varrho)$ is the smallest non-zero category that one could expect (this is the most obvious in the case that S is a point). This agrees with the intuition that the blow-up resolution is minimal in some way.

Question 5.6. Given a variety Y with rational Gorenstein singularities, is there a resolution $\varrho: \tilde{Y} \rightarrow Y$ of minimal categorical discrepancy in the sense that, for every other resolution $\varrho': \tilde{Y}' \rightarrow Y$, there is a fully faithful embedding $\text{disc}(\varrho) \hookrightarrow \text{disc}(\varrho')$?

Often, in the case of a quotient singularity, a good candidate for a resolution of minimal categorical discrepancy should be the G -Hilbert scheme.

At least, we can see that $\text{disc}(\varrho)$ grows if we further blow-up the resolution.

Proposition 5.7. *Let $\varrho: \tilde{Y} \rightarrow Y$ be a resolution of singularities and let $f: \tilde{Y}' \rightarrow \tilde{Y}$ be the blow-up in a smooth center $C \subset \tilde{Y}$. Set $\varrho' := \varrho f: \tilde{Y}' \rightarrow Y$. Then there is a semi-orthogonal decomposition*

$$\text{disc}(\varrho') = \langle \mathcal{R}, \text{disc}(\varrho) \rangle \quad \text{with } \mathcal{R} \neq 0.$$

Proof. We have a semi-orthogonal decomposition $D^b(\tilde{Y}') = \langle \mathcal{A}, \mathcal{B} \rangle$ with $\mathcal{B} = f^* D^b(\tilde{Y})$ and

$$\mathcal{A} = \langle \iota_*(g^* D^b(C) \otimes \mathcal{O}_g(-c+1)), \dots, \iota_*(g^* D^b(C) \otimes \mathcal{O}_g(-1)) \rangle.$$

Here, $c = \text{codim}(C \hookrightarrow \tilde{Y})$ and g and ι are the \mathbb{P}^{n-1} -bundle projection and the inclusion of the exceptional divisor of the blow-up. We get a semi-orthogonal decomposition

$$\text{disc}(\varrho') = \langle \mathcal{A}/(\text{WCN}(\varrho') \cap \mathcal{A}), \mathcal{B}/(\text{WCN}(\varrho') \cap \mathcal{B}) \rangle;$$

this can be deduced from [LS16, Prop. B.2]. We have $f_* \mathcal{O}_{\tilde{Y}'} \cong \mathcal{O}_{\tilde{Y}} \cong f_* \omega_f$. By projection formula, it follows that $f_{*|\mathcal{B}} \cong f_{!|\mathcal{B}}$. Hence, we have $\mathcal{B} \cap \text{WCN}(\varrho') = f^* \text{WCN}(\varrho)$ and $\mathcal{B}/(\text{WCN}(\varrho') \cap \mathcal{B}) \cong \text{disc}(\varrho)$. Thus, to get the assertion, it is sufficient to prove that $\text{WCN}(\varrho') \cap \mathcal{A}$ is a proper subcategory of \mathcal{A} . Note that $f_*(\mathcal{A}) = 0$, hence $\varrho'_*(\mathcal{A}) = 0$ and $\text{WCN}(\varrho') \cap \mathcal{B} = \ker(\varrho'_!) \cap \mathcal{A}$. Let $x \in C$ be a point and $E_x \subset \tilde{Y}$ the fibre of f over x . We consider

$$A := \mathcal{O}_{E_x}(-c+1) \in \iota_*(g^* D^b(C) \otimes \mathcal{O}_g(-c+1)) \subset \mathcal{A}.$$

We have $f_! A \cong f_* \mathcal{O}_{E_x} \cong \mathcal{O}_x$. Hence, $\varrho'_! A \cong \varrho_*(\mathcal{O}_x) \neq 0$. □

Remark 5.8. One can show that the category \mathcal{R} is always a quotient of $D^b(C)$.

5.4. (Non-)unicity of categorical crepant resolutions. Let $\tilde{Y} \rightarrow Y$ be a resolution of rational Gorenstein singularities and let $\mathcal{D} \subset \mathrm{D}^{\mathrm{perf}}(\tilde{Y})$ be an admissible subcategory which is a weakly crepant resolution, i.e. $\varrho^*: \mathrm{D}^{\mathrm{perf}}(Y) \rightarrow \mathrm{D}^{\mathrm{b}}(\tilde{Y})$ factors through \mathcal{D} and $\varrho_{*|\mathcal{D}} \cong \varrho_{!!|\mathcal{D}}$. Then every admissible subcategory $\mathcal{D}' \subset \mathcal{D}$ with the property that $\varrho^*: \mathrm{D}^{\mathrm{perf}}(Y) \rightarrow \mathrm{D}^{\mathrm{b}}(\tilde{Y})$ factors through \mathcal{D}' is a weakly crepant resolution too.

In particular, in our setup of cyclic quotients, there is a tower of weakly crepant resolutions of length $n - m$ given by successively dropping the $\mathrm{D}^{\mathrm{b}}(S)$ parts of the semi-orthogonal decomposition of $\mathrm{WCN}(\varrho)$. We see that weakly crepant categorical resolutions are not unique, even if we fix the ambient derived category $\mathrm{D}^{\mathrm{b}}(\tilde{Y})$ of a geometric resolution $\tilde{Y} \rightarrow Y$.

In contrast, *strongly crepant* categorical resolutions are expected to be unique up to equivalence; see [Kuz08, Conj. 4.10]. A strongly crepant categorical resolution of $\mathrm{D}^{\mathrm{b}}(Y)$ is a module category over $\mathrm{D}^{\mathrm{b}}(Y)$ with trivial relative Serre functor; see [Kuz08, Sect. 3]. For an admissible subcategory $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(\tilde{Y})$ of the derived category of a geometric resolution of singularities $\varrho: \tilde{Y} \rightarrow Y$ this condition means that \mathcal{D} is Y -linear and there are functorial isomorphisms

$$(14) \quad \varrho_* \mathcal{H}om(A, B)^\vee \cong \varrho_* \mathcal{H}om(B, A)$$

for $A, B \in \mathcal{D}$. In our cyclic setup, $\Psi(\mathrm{D}_G(X)) \subset \mathrm{D}^{\mathrm{b}}(\tilde{Y})$ is a strongly crepant categorical resolution; see [Kuz08, Thm. 1] or [Abu16, Thm. 10.2].

We require strongly crepant categorical resolutions to be *indecomposable* which means that they do not decompose into direct sums of triangulated categories or, in other words, they do not admit both-sided orthogonal decompositions. Under this additional assumption, we can prove that strongly crepant categorical resolutions are unique if we fix the ambient derived category of a geometric resolution.

Proposition 5.9. *Let $\tilde{Y} \rightarrow Y$ be a resolution of Gorenstein singularities and $\mathcal{D}, \mathcal{D}' \subset \mathrm{D}^{\mathrm{b}}(\tilde{Y})$ admissible indecomposable strongly crepant subcategories. Then $\mathcal{D} = \mathcal{D}'$.*

Proof. The intersection $\mathcal{D} \cap \mathcal{D}'$ is again an admissible Y -linear subcategory of $\mathrm{D}^{\mathrm{b}}(\tilde{Y})$ containing $\varrho^*(\mathrm{D}^{\mathrm{perf}}(Y))$. Furthermore, condition (14) is satisfied for every pair of objects of $\mathcal{D} \cap \mathcal{D}'$ so that the intersection is again a strongly crepant resolution. Hence, we can assume without loss of generality that $\mathcal{D}' \subset \mathcal{D}$.

Let \mathcal{A} be the right-orthogonal complement of \mathcal{D}' in \mathcal{D} so that we have a semi-orthogonal decomposition $\mathcal{D} = \langle \mathcal{A}, \mathcal{D}' \rangle$. By Lemma 2.6, this means that $\varrho_* \mathcal{H}om(D, A) = 0$ for $A \in \mathcal{A}$ and $D \in \mathcal{D}'$. But then, by (14), we also get $\varrho_* \mathcal{H}om(A, D) = 0$ so that $\mathcal{D} = \mathcal{A} \oplus \mathcal{D}'$. \square

5.5. Connection to Calabi–Yau neighbourhoods. In [HKP12], *spherelike objects* and *spherical subcategories* generated by them were introduced and studied. The paper gave some evidence that these objects and subcategories might play a role in birationality questions for Calabi–Yau varieties. One of the starting points for our project was the idea to consider spherical subcategories, and their generalisations *Calabi–Yau neighbourhoods*, as candidates for categorical crepant resolutions of Calabi–Yau quotient varieties. In this subsection, we briefly describe the connection to the weakly crepant resolutions considered above.

We recall some abstract homological notions. Let \mathcal{T} be a Hom-finite \mathbb{C} -linear triangulated category and $E \in \mathcal{T}$ an object. We say that $SE \in \mathcal{T}$ is a *Serre dual object* for E if the functors $\mathrm{Hom}^*(E, -)$ and $\mathrm{Hom}^*(-, SE)^\vee$ are isomorphic. By the Yoneda lemma, SE is then uniquely determined. Fix an integer d . We call the object E

- a *d-Calabi–Yau object*, if $E[d]$ is a Serre dual of E ,

- *d-spherelike* if $\mathrm{Hom}^*(E, E) = \mathbb{C} \oplus \mathbb{C}[-d]$, and
- *d-spherical* if E is *d-spherelike* and a *d-Calabi–Yau* object.

Note a smooth compact variety X of dimension d is a strict Calabi–Yau variety precisely if the structure sheaf \mathcal{O}_X is a *d-spherical* object of $\mathrm{D}^b(X)$.

In [HKP12] the authors show that if E is a *d-spherelike* object, there exists a unique maximal triangulated subcategory of \mathcal{T} in which E becomes *d-spherical*. In the following we will imitate this construction for a larger class of objects.

Definition 5.10. Let $E \in \mathcal{T}$ be an object in a triangulated category having a Serre dual SE . We call E a *d-selfdual object* if

- (i) $\mathrm{Hom}(E, E[d]) \cong \mathbb{C}$, i.e. by Serre duality there is a morphism $w: E \rightarrow \omega(E) := SE[-d]$ unique up to scalars, and
- (ii) the induced map $w_*: \mathrm{Hom}^*(E, E) \xrightarrow{\sim} \mathrm{Hom}^*(E, \omega(E))$ is an isomorphism.

In particular, a *d-selfdual* object satisfies $\mathrm{Hom}^*(E, E) \cong \mathrm{Hom}(E, E)^\vee[-d]$, hence the name.

Remark 5.11. If an object is *d-spherelike*, then it is *d-selfdual*; compare [HKP12, Lem. 4.2].

For a *d-selfdual* object E , there is a triangle $E \rightarrow \omega(E) \rightarrow Q_E \rightarrow E[1]$ induced by w . By our assumption, we get $\mathrm{Hom}^*(E, Q_E) = 0$. Thus, following [HKP12], we propose the following

Definition 5.12. The *Calabi–Yau neighbourhood* of a *d-selfdual* object $E \in \mathcal{T}$ is the full triangulated subcategory

$$\mathrm{CY}(E) := {}^\perp Q_E \subseteq \mathcal{T}.$$

Proposition 5.13. If $E \in \mathcal{T}$ is a *d-selfdual* object then $E \in \mathrm{CY}(E)$ is a *d-Calabi–Yau* object.

Proof. If $T \in \mathrm{CY}(E)$, apply $\mathrm{Hom}^*(T, -)$ to the triangle $E \rightarrow \omega(E) \rightarrow Q_E$. \square

Using the same proof as for [HKP12, Thm. 4.6], we see that the Calabi–Yau neighbourhood is the maximal subcategory of \mathcal{T} in which a *d-selfdual* object E becomes *d-Calabi–Yau*.

Proposition 5.14. If $\mathcal{U} \subset \mathcal{T}$ is a full triangulated subcategory and $E \in \mathcal{U}$ is *d-Calabi–Yau*, then $\mathcal{U} \subset \mathrm{CY}(E)$.

Proposition 5.15. Let Y be a projective variety with rational Gorenstein singularities and trivial canonical bundle of dimension $d = \dim Y$ and consider a resolution of singularities $\varrho: \tilde{Y} \rightarrow Y$. Then, for every line bundle $L \in \mathrm{Pic} Y$, the pull-back $\varrho^*L \in \mathrm{D}^b(\tilde{Y})$ is *d-selfdual*. Furthermore, we have

$$(15) \quad \mathrm{WCN}(\varrho) = \bigcap_{L \in \mathrm{Pic} Y} \mathrm{CY}(\varrho^*L).$$

Proof. Note that, by our assumption that ω_Y is trivial, we have $\omega_{\tilde{Y}} \cong \omega_{\varrho}$. Hence, by Grothendieck duality, there is a morphism $w_L: \varrho^*L \rightarrow \varrho^*L \otimes \omega_{\tilde{Y}}$ unique up to scalar multiplication, namely $w_L = \mathrm{id}_{\varrho^*L} \otimes s$ where s is the non-zero section of $\omega_{\tilde{Y}} \cong \omega_{\varrho}$; compare the previous Subsection 5.1. Furthermore, $w_{L*}: \mathrm{Hom}^*(\varrho^*L, \varrho^*L) \rightarrow \mathrm{Hom}^*(\varrho^*L, \varrho^*L \otimes \omega_{\tilde{Y}})$ is an isomorphism, still by Grothendieck duality, which means that ϱ^*L is *d-selfdual*.

Recall that $\mathrm{WCN}(\varrho) = \ker(\varrho_+)$ where ϱ_+ is defined as the cone

$$\varrho_* \xrightarrow{t} \varrho! \rightarrow \varrho_+ \rightarrow .$$

By adjunction, we get $\mathrm{WCN}(\varrho) = {}^\perp(\varrho^+(\mathrm{D}^{\mathrm{perf}}(Y)))$ where $\varrho^+ = \varrho_+^R$ is given by the triangle

$$\varrho^+ \rightarrow \varrho^* \xrightarrow{t^R} \varrho^! \rightarrow .$$

Note that $t^R = (_)\otimes s$. Hence $t^R(L) = w_L: \varrho^*L \rightarrow \varrho^*L \otimes \omega_{\tilde{Y}}$ and $\varrho^+(L) = Q_{\varrho^*L}[-1]$; compare [Definition 5.12](#). Since the line bundles form a generator of $D^{\text{perf}}(Y)$, we get for $F \in D^b(\tilde{Y})$:

$$\begin{aligned} F \in \text{WCN}(\varrho) &\iff F \in {}^\perp(\varrho^+(D^{\text{perf}}(Y))) \\ &\iff F \in {}^\perp Q_{\varrho^*L} \quad \forall L \in \text{Pic } Y \\ &\iff F \in \text{CY}(\varrho^*L) \quad \forall L \in \text{Pic } Y. \end{aligned} \quad \square$$

Remark 5.16. Following the proof of [Proposition 5.15](#), we see that, on the right-hand side of (15), it is sufficient to take the intersection over all powers of a given ample line bundle.

In our cyclic setup, if S consists of isolated points, we even have $\text{WCN}(\varrho) = \text{CY}(\mathcal{O}_{\tilde{Y}})$ so that the weakly crepant neighbourhood is computed by a Calabi-Yau neighbourhood of a single object. The same should hold in general if Y has isolated singularities.

6. STABILITY CONDITIONS FOR KUMMER THREEFOLDS

Let A be an abelian variety of dimension g . Consider the action of $G = \mu_2$ by ± 1 . Then the fixed point set $A[2]$ consists of the 4^g two-torsion points. Consider the quotient \overline{A} (the *singular Kummer variety*) of A by G , and the blow-up $K(A)$ (the *Kummer resolution*) of \overline{A} in $A[2]$. This setup satisfies [Condition 3.1](#), with $m = 2$ and $n = g$ and we get

Corollary 6.1. *The functor $\Psi: D_G^b(A) \rightarrow D^b(K(A))$ is fully faithful, and*

$$D^b(K(A)) = \left\langle \underbrace{D^b(\text{pt}), \dots, D^b(\text{pt})}_{(g-2)4^g \text{ times}}, \Psi(D_G^b(A)) \right\rangle.$$

To explore a potentially useful consequence of this result, we need to recall that a Bridgeland stability condition on a reasonable \mathbb{C} -linear triangulated category \mathcal{D} consists of the heart \mathcal{A} of a bounded t-structure in \mathcal{D} and a function from the numerical Grothendieck group of \mathcal{D} to the complex numbers satisfying some axioms, see [\[Bri07\]](#).

Corollary 6.2. *There exists a Bridgeland stability condition on $D^b(K(A))$, for an abelian threefold A .*

Proof. To begin with, one checks that $D_G^b(A)$ has a stability condition, which follows quite easily from [\[BMS16, Cor. 10.3\]](#). For a two-torsion point $x \in A[2]$, we set $E_x := \mathcal{O}_{\varrho^{-1}(\pi(x))}(-1)$. Then, since $g = \dim A = 3$, the semi-orthogonal decomposition of [Corollary 6.1](#) is given by

$$(16) \quad D^b(K(A)) = \langle \{E_x\}_{x \in A[2]}, D_G^b(A) \rangle.$$

Next, we want to show that, for every $x \in A[2]$, there exists an integer i such that $\text{Hom}^{\leq i}(E_x, \Psi(F)) = 0$ for all $F \in \mathcal{A} \subset D_G^b(A)$. Indeed, the cohomology of any complex in the heart of the stability condition on $D_G^b(A)$, as constructed in [\[BMS16, Cor. 10.3\]](#), is concentrated in an interval of length three. The functor Ψ has cohomological amplitude at most 3, since $q_*^G: \text{Coh}_G(A) \rightarrow \text{Coh}(K(A))$ is an exact functor of abelian categories, and every sheaf on A has a locally free resolution of length $\dim A = 3$. This implies that the cohomology of any complex in $\Psi(D_G^b(A))$ is contained in a fixed interval of length 6. This proves the above claim. Using [\[CP10, Prop. 3.5\(b\)\]](#), this then implies that $\langle E_x, \Psi(D_G^b(A)) \rangle$ has a stability condition; compare the argument in [\[BMMS12, Cor. 3.8\]](#).

We can proceed to show that, for $x \neq y \in A[2]$, there exists an integer i such that $\text{Hom}^{\leq i}(E_y, \langle E_x, \Psi(D_G^b(A)) \rangle) = 0$ and hence there is a stability condition on $\langle E_y, E_x, \Psi(D_G^b(A)) \rangle$. After 4^3 steps we have constructed a stability condition on $D^b(K(A))$; compare (16). \square

REFERENCES

- [Abu16] Roland Abuaf. Categorical crepant resolutions for quotient singularities. *Math. Z.*, 282:679–689, 2016.
- [AC12] Dima Arinkin and Andrei Căldăraru. When is the self-intersection of a subvariety a fibration? *Adv. Math.*, 231(2):815–842, 2012.
- [ACH14] Dima Arinkin, Andrei Căldăraru, and Marton Hablicsek. Formality of derived intersections and the orbifold HKR isomorphism. *arXiv:1412.5233*, 2014.
- [ADM15] Nicolas Addington, Will Donovan, and Ciaran Meachan. Mukai flops and \mathbb{P} -twists. *arXiv:1507.02595*, 2015. To appear in *Crelle*.
- [AL12] Rina Anno and Timothy Logvinenko. On adjunctions for Fourier–Mukai transforms. *Adv. Math.*, 231(3-4):2069–2115, 2012.
- [AL13] Rina Anno and Timothy Logvinenko. Spherical dg-functors. *arXiv:1309.5035*, 2013.
- [BKR01] Tom Bridgeland, Alastair King, and Miles Reid. The McKay correspondence as an equivalence of derived categories. *J. Amer. Math. Soc.*, 14(3):535–554 (electronic), 2001.
- [BMMS12] Marcello Bernardara, Emanuele Macrì, Sukhendu Mehrotra, and Paolo Stellari. A categorical invariant for cubic threefolds. *Adv. Math.*, 229:770–803, 2012.
- [BMS16] Arend Bayer, Emanuele Macrì, and Paolo Stellari. The space of stability conditions on abelian threefolds, and on some Calabi-Yau threefolds. *Invent. Math.*, 206:869–933, 2016.
- [BO95] Alexei Bondal and Dmitri Orlov. Semiorthogonal decomposition for algebraic varieties. *arXiv:alg-geom/9506012*, 1995.
- [Bon89] Alexei Bondal. Representations of associative algebras and coherent sheaves. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53(1):25–44, 1989.
- [Bri02] Tom Bridgeland. Flops and derived categories. *Invent. Math.*, 147(3):613–632, 2002.
- [Bri07] Tom Bridgeland. Stability conditions on triangulated categories. *Ann. Math.*, 166:317–346, 2007.
- [BS10] Tom Bridgeland and David Stern. Helices on del Pezzo surfaces and tilting Calabi-Yau algebras. *Adv. Math.*, 224:1672–1716, 2010.
- [Car57] Henri Cartan. Quotient d’un espace analytique par un groupe d’automorphismes. In *Algebraic Geometry and Algebraic Topology, a Symposium in Honor of S. Lefschetz*, pages 90–102. Princeton University Press, Princeton, NJ, 1957.
- [CP10] John Collins and Alexander Polishchuk. Gluing stability conditions. *Adv. Theor. Math. Phys.*, 14(2):563–607, 2010.
- [DW15] Will Donovan and Michael Wemyss. Twists and braids for general 3-fold flops. *arXiv:1504.05320*, 2015.
- [DW16] Will Donovan and Michael Wemyss. Noncommutative deformations and flops. *Duke Math. J.*, 165(8):1397–1474, 2016.
- [Ef14] Alexander I. Efimov. Derived categories of Grassmannians over integers and modular representation theory. *arXiv:1410.7462v2*, 2014.
- [Göt94] Lothar Göttsche. *Hilbert schemes of zero-dimensional subschemes of smooth varieties*, volume 1572 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1994.
- [Gri13] Julien Grivaux. Formality of derived intersections. *arXiv:1302.0237*, 2013.
- [Har80] Robin Hartshorne. Stable reflexive sheaves. *Math. Ann.*, 254(2):121–176, 1980.
- [HKP12] Andreas Hochenegger, Martin Kalck, and David Ploog. Spherical subcategories. *arXiv:1208.4046*, 2012.
- [Huy06] Daniel Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, Oxford, 2006.
- [IU15] Akira Ishii and Kazushi Ueda. The special McKay correspondence and exceptional collections. *Tohoku Math. J. (2)*, 67(4):585–609, 2015.
- [Kaw02] Yujiro Kawamata. D -equivalence and K -equivalence. *J. Differential Geom.*, 61(1):147–171, 2002.
- [Kaw16] Yujiro Kawamata. Derived McKay correspondence for $\mathrm{GL}(3, \mathbb{C})$. *arXiv:1609.09540 [math.AG]*, 2016.
- [KL15] Alexander Kuznetsov and Valery A. Lunts. Categorical resolutions of irrational singularities. *Int. Math. Res. Not. IMRN*, (13):4536–4625, 2015.
- [Kov00] Sándor J. Kovács. A characterization of rational singularities. *Duke Math. J.*, 102(2):187–191, 2000.

- [KP14] Alexander Kuznetsov and Alexander Perry. Derived categories of cyclic covers and their branch divisors. *arXiv:1411.1799*, 2014.
- [Kru14] Andreas Krug. P-functor versions of the Nakajima operators. *arXiv:1405.1006*, 2014.
- [Kuz06] Alexander Kuznetsov. Hyperplane sections and derived categories. *Izv. Ross. Akad. Nauk Ser. Mat.*, 70(3):23–128, 2006.
- [Kuz08] Alexander Kuznetsov. Lefschetz decompositions and categorical resolutions of singularities. *Selecta Math. (N.S.)*, 13(4):661–696, 2008.
- [Lim16] Bronson Lim. Equivariant derived categories associated to a sum of two potentials. *arXiv:1611.03058*, 2016.
- [LS16] Valery Lunts and Olaf Schnürer. Matrix factorizations and semi-orthogonal decompositions for blowing-ups. *J. Noncommut. Geom.*, 10(3):907–979, 2016.
- [Nam03] Yoshinori Namikawa. Mukai flops and derived categories. *J. Reine Angew. Math.*, 560:65–76, 2003.
- [Nee01] Amnon Neeman. *Triangulated categories*, volume 148 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001.
- [Orl93] Dmitri Orlov. Projective bundles, monoidal transformations, and derived categories of coherent sheaves. *Izv. Math.*, 41:133–141, 1993.
- [Orl09] Dmitri Orlov. Remarks on generators and dimensions of triangulated categories. *Moscow Math. J.*, 9:153–159, 2009.
- [Plo07] David Ploog. Equivariant autoequivalences for finite group actions. *Adv. Math.*, 216(1):62–74, 2007.
- [PVdB15] Alexander Polishchuk and Michel Van den Bergh. Semiorthogonal decompositions of the categories of equivariant coherent sheaves for some reflection groups. *arXiv:1503.04160*, 2015.
- [Sca15] Luca Scala. Notes on diagonals of the product and symmetric variety of a surface. *arXiv:1510.04889*, 2015.
- [Tod07] Yukinobu Toda. On a certain generalization of spherical twists. *Bull. Soc. Math. France*, 135(1):119–134, 2007.

Contact:

- A. K.: Philipps-Universität Marburg, Hans-Meerwein-Straße 6, Campus Lahnberge, 35032 Marburg, Germany
Email: andkrug@mathematik.uni-marburg.de
- D. P.: Freie Universität Berlin, Mathematisches Institut, Arnimallee 3, 14195 Berlin, Germany
Email dploog@math.fu-berlin.de
- P. S.: Universität Hamburg, Fachbereich Mathematik, Bundesstraße 55, 20146 Hamburg, Germany
Email pawel.sosna@math.uni-hamburg.de

$$\begin{array}{ccc}
 \tilde{X} & \xrightarrow[p]{\text{blow-up in } S} & X \\
 \downarrow q & \swarrow j & \searrow a \\
 & Z = \mathbb{P}(N) \xrightarrow{\nu} S & \\
 & \swarrow i & \searrow b \\
 \tilde{X}/G = \tilde{Y} & \xrightarrow[\varrho]{\text{blow-up in } S} & Y = X/G \\
 \downarrow \Xi_\alpha & & \downarrow \Theta_\beta \\
 D^b(S) & & D^b(X)
 \end{array}$$

$$\begin{array}{ccc}
 D^b(\tilde{Y}) & \xrightleftharpoons[\Psi]{\Phi} & D_G^b(X) \\
 \uparrow \Xi_\alpha & & \uparrow \Theta_\beta \\
 D^b(S) & & D^b(X)
 \end{array}$$

$G = \mu_m = \langle g \rangle$ acts on smooth X
 $S = \text{Fix}(G) \subset X, n = \dim(X) - \dim(S)$
 $N = N_{S/X}$ with $g|_N = \zeta \cdot \text{id}_N$
 $\chi: G \rightarrow \mathbb{C}^*, \chi(g) = \zeta^{-1}$

$\mathcal{L}_{\tilde{Y}} \in \text{Pic}(\tilde{Y})$ with $\mathcal{L}_{\tilde{X}}^m = \mathcal{O}_{\tilde{Y}}(Z)$
 $\mathcal{L}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(Z) \in \text{Pic}^G(\tilde{X})$ with trivial
 action on $\mathcal{L}_{\tilde{X}|Z} = \mathcal{O}_Z(Z) = \mathcal{O}_\nu(-1)$

$\Phi := p_* \circ q^* \circ \text{triv}$
 $\Psi := (-)^G \circ q_* \circ p^*$
 $\Theta_\beta := i_*(\nu^*(_) \otimes \mathcal{O}_\nu(\beta))$
 $\Xi_\alpha := (a_* \circ \text{triv}) \otimes \chi^\alpha$